Infinite Series Review

Math 12

It is complicated to determine whether a given series $\sum a_n$ converges. Some of the main difficulties for deciding convergence, are maybe deciding which convergence test to use, how to apply it correctly, and finally, how to make the correct conclusion based on the results of your test(s). Here are some basic guidelines:

- **Geometric Series Test**
  - If the series is **geometric** like $\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$,

    then
    1. if $|r| < 1$, it converges to $\dfrac{a}{1-r}$
    2. if $|r| \geq 1$, it diverges.

    USES: Only for geometric series. This is one of the few times that you can actually compute the sum of the series, rather than just make a conclusion about convergence. Often used together with one of the comparison tests.

    RESTRICTIONS: Only works for Geometric Series.

- **p-Series Test or p-Test**
  - If the series is a **p-series** of the form $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$,

    then
    1. if $p > 1$, it converges
    2. if $p \leq 1$ it diverges.

    USES: Only for p-series. Most commonly used together with one of the comparison tests. Proved using the Integral Test.

    RESTRICTIONS: Only for series exactly of this form $\sum \dfrac{1}{n^p}$.

- **n^{th} Term Divergence Test**
  - Consider a series $\sum_{n=1}^{\infty} a_n$, then
    1. if $\lim_{n\to\infty} a_n \neq 0$ or DNE, then it diverges
    2. if $\lim_{n\to\infty} a_n = 0$ the test is **inconclusive**.

    USES: If you are not sure how to approach a series, this is a good test to try first. Or if you notice that the terms just don't approach 0, apply this test and your work here will be done.

    RESTRICTIONS: Only helpful for series whose terms do **not** go to zero.

- **(Direct) Comparison Test**
  - Consider a series $\sum a_n$. First find a series $\sum b_n$ with which to compare $\sum a_n$. Suppose that both $\sum a_n$ and $\sum b_n$ have positive terms. Then
    1. if $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
    2. if $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

    USES: To reduce a meaty fraction term and compare directly to a nicer form, as geometric or p series. The idea here is a_n is say *better* (maybe a lot better) than b_n in terms of size, or a_n is *worse* than b_n in terms of size. Best for bounded pieces like $\sin^2 n \leq 1$, $\ln n < n$, $\ln n < \sqrt{n}$, or $\arctan n < \frac{\pi}{2}$. Or good when you have a direct and helpful size comparison at hand.

    RESTRICTIONS: Only used for positive term series. Be careful about which way the size implications go.

- **Limit Comparison Test**
  - Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

    $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

    where c is **finite**, $0 < c < \infty$, then either **both** series converge or both diverge.

    USES: To reduce a meaty fraction term and compare to a nicer form, as geometric or p series. The idea here is $\sum a_n$ and $\sum b_n$ share the **same** behavior. Best for fractional terms like $a_n = \dfrac{\text{polynomial}}{\text{polynomial}}$. Examine dominant terms of the numerator or denominator when deciding which series to compare with. Use LCT to eliminate extraneous terms (lower orders of magnitude).

    RESTRICTIONS: Only used for positive term series.

- **Ratio Test**
  - Consider a series $\sum_{n=1}^{\infty} a_n$. Then
    1. if $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is **absolutely** convergent.
    2. if $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
    3. if $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is **inconclusive**.

    USES: Mostly useful for series involving exponentials (constants raised to powers or n^n) or factorials, or combinations of those. If your series already has strictly positive terms, then absolute convergence is just convergence. If your series does not have strictly positive terms, then absolute convergence implies convergence. This result is strong in

the sense that it gives you divergence of the original series $\sum_{n=1}^{\infty} a_n$, not just $\sum_{n=1}^{\infty} |a_n|$. Not helpful when your terms contain just polynomials or just natural logs.
RESTRICTIONS: The limit L must exist or equal ∞, and it must not equal 1.

- **Root Test** (not on this Exam #2)
  - Consider a series $\sum_{n=1}^{\infty} a_n$. Then
    1. if $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ is **absolutely** convergent.
    2. if $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
    3. if $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is **inconclusive**.

    USES: Mostly for series involving complicated things raised to the power of n. But in most cases where the root test can be used, the ratio test can also be used.
    RESTRICTIONS: The limit L must exist or equal ∞, and it must not equal 1.

- **Absolute Convergence Test**
  - Consider a series $\sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
    That is, absolute convergence implies convergence.
    USES: Useful if the absolute series $\sum_{n=1}^{\infty} |a_n|$ is easier to examine. Helps you sometimes avoid using the Alternating Series test (see below). Might answer a question about Absolute Convergence immediately.
    RESTRICTIONS: Doesn't completely help you answer a question about conditional convergence, since if $\sum_{n=1}^{\infty} |a_n|$ diverges, then we know nothing about the original $\sum_{n=1}^{\infty} a_n$. This is slightly different from the divergence conclusion in the Ratio Test.

- **Alternating Series Test**
  - Consider a series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with terms alternating in sign. Then if the following three conditions are all satisfied:
    1. if each $b_n > 0$
    2. if $\lim_{n\to\infty} b_n = 0$
    3. if $b_{n+1} \le b_n$,

    then the series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges.

USES: Given an alternating series, use this, UNLESS the alternating series is quickly examined using the Ratio Test or as the related absolute series with the Absolute Convergence Test. If the absolute series $\sum_{n=1}^{\infty} \left|(-1)^{n+1} b_n\right|$, which equals $\sum_{n=1}^{\infty} b_n$, converges, then you automatically know that the original series itself converges, by the Absolute Convergence test. When you do use AST, check condition 2; if it fails, the series already was diverging by the n^{th} term divergence test. Condition 3 is often verified by an obvious size argument or by taking the derivative of the related function and seeing that it is negative, and hence the terms are decreasing.

RESTRICTIONS: For alternating series only.

- **Integral Test**

  - Consider a series $\sum_{n=1}^{\infty} a_n$. Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. That is,

    1. if $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum a_n$ is convergent.
    2. if $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum a_n$ is divergent.

    USES: Used to prove the p-Test, but once you have that, it isn't used that often. Often used when no other test works. Used on series like $\sum \frac{1}{n \ln n}$ or $\sum \frac{1}{n(\ln n)^p}$. You can remember the result by thinking that the related improper integral and the series share the same behavior, both converge or both diverge. This test can be used on many series, but often those series can be examined more easily with a different inspecting test.

    RESTRICTIONS: Only works for positive, decreasing series.

- **Note** The above quidelines are not listed in any particular order. Be familiar with the conditions and conclusions for each test. Unfortunately, they are all very similar looking. It may also take more than one try at a test to figure out a given series. Sometimes it even takes a combination of tests to make a final conclusion about convergence.

  Think of these tests the same way as say the TB test at the doctor's office. If you want to know if you have TB, take the seemingly irrelevant TB test (and if doesn't seem irrelevant to you, try explaining to a four-year-old why getting a puncture in the arm tells whether you have a disease of the lungs). In the same way, if you want to know whether a series converges, apply one of the seemingly irrelevant convergence test. The remaining problem is, you have to figure out which test to use; for a given problem, probably only a few tests are even applicable, and maybe only one will actually give you a conclusive result.