Math 12 Helpful Sequence Tips and Facts

- $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
- $\lim_{n\to\infty} \frac{\ln n}{n^r} = 0 \quad \text{for } r > 0$
- $\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{x} = 1 \qquad \text{for a fixed constant number } x$
- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad \text{for a fixed constant number } x$
- $\lim_{n\to\infty} \frac{n!}{n^n} = 0$
- $\lim_{n\to\infty} \frac{x^n}{n^n} = 0 \qquad \text{for a fixed constant number } x$
- $\lim_{n\to\infty} \frac{e^n}{n^r} = \infty \qquad \text{for some fixed power } r > 0$
- $$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 & \text{think } \left(\frac{1}{2}\right)^n \to 0 \quad \text{as} \quad n \to \infty \\ 1 & \text{if } r = 1 & \text{think } 1^n \to 1 \quad \text{as} \quad n \to \infty \\ \text{Diverges} & \text{if } |r| > 1 & \text{think} \begin{cases} 7^n \to \infty \quad \text{as} \quad n \to \infty \\ (-7)^n \text{ diverges} \quad \text{as} \quad n \to \infty \end{cases} \\ \text{Diverges} & \text{if } r = -1 & \text{think } (-1)^n \text{ diverges} \quad \text{as} \quad n \to \infty \end{cases}$$

A helpful and general summary is, as $n \to \infty$,

$\ln n$	$\langle\langle$	n^r	$(r > 0)$	$\langle\langle$	a^n	$(a > 1)$	$\langle\langle$	$n!$	$\langle\langle$	n^n

Here $\langle\langle$ represents the notion of being *much smaller in size.* That is, as n grows large, size-wise,

logs	$\langle\langle$	ploynomials	$\langle\langle$	exponentials	$\langle\langle$	factorials	$\langle\langle$	"super" exponentials

Proof of a few facts:

- To prove $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$, we consider the related function $f(x) = x^{\frac{1}{x}}$, in order to apply L'H Rule. Then

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln\left(x^{\frac{1}{x}}\right)} = e^{\left(\lim\limits_{x\to\infty} \ln\left(x^{\frac{1}{x}}\right)\right)} = e^{\left(\lim\limits_{x\to\infty} \frac{\ln x}{x}\right)^{\left(\frac{\infty}{\infty}\right)}} \overset{\text{L'H}}{=} e^{\left(\lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1}\right)} = e^0 = 1$$

- To prove $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1$, we consider the related function $f(t) = x^{\frac{1}{t}}$. For a fixed x, note that $\ln x$ is a fixed constant. Then,

$$\lim_{t\to\infty} x^{\frac{1}{t}} = \lim_{t\to\infty} e^{\ln\left(x^{\frac{1}{t}}\right)} = e^{\left(\lim\limits_{t\to\infty} \ln\left(x^{\frac{1}{t}}\right)\right)} = e^{\left(\lim\limits_{t\to\infty} \frac{\ln x}{t}\right)} = e^0 = 1$$

- To prove $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$, we consider the related function $f(x) = \left(1 + \frac{1}{x}\right)^x$. Then,

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{\left(\lim\limits_{x\to\infty} \ln\left(1 + \frac{1}{x}\right)^x\right)} = e^{\left(\lim\limits_{x\to\infty} x\ln\left(1 + \frac{1}{x}\right)\right)}$$

$$\overset{\infty\cdot 0}{=} e^{\left(\lim\limits_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right)^{\left(\frac{0}{0}\right)}} \overset{\text{L'H}}{=} e^{\left(\lim\limits_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}\right)} = e^{\left(\lim\limits_{x\to\infty} \frac{1}{1+\frac{1}{x}}\right)} = e^1 = e$$

- To prove $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$, we consider the related function $f(t) = \left(1 + \frac{x}{t}\right)^t$, for a fixed number x. Then,

$$\lim_{t\to\infty} \left(1 + \frac{x}{t}\right)^t = \lim_{t\to\infty} e^{\ln\left(1+\frac{x}{t}\right)^t} = e^{\left(\lim\limits_{t\to\infty} \ln\left(1 + \frac{x}{t}\right)^t\right)} = e^{\left(\lim\limits_{t\to\infty} t\ln\left(1 + \frac{x}{t}\right)\right)}$$

$$\overset{\infty\cdot 0}{=\!=} e^{\left(\lim_{t\to\infty} \frac{\ln\left(1+\frac{x}{t}\right)}{\frac{1}{t}}\right)} \overset{\text{L'H}}{=} e^{\left(\lim_{t\to\infty} \frac{\frac{1}{1+\frac{x}{t}}\cdot\left(-\frac{x}{t^2}\right)}{-\frac{1}{t^2}}\right)} = e^{\left(\lim_{t\to\infty} \frac{x}{1+\frac{x}{t}}\right)} = e^x$$

- To prove $\lim_{n\to\infty} \frac{n!}{n^n} = 0$, we work to find a bound on $\frac{n!}{n^n}$ for n large.

$$\begin{aligned}
\frac{n!}{n^n} &= \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\cdot 4\cdot 3\cdot 2\cdot 1}{n\cdot n\cdot n\cdot n\cdot\ldots\cdot n\cdot n\cdot n\cdot n} \\
&= \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\left(\frac{n-3}{n}\right)\cdots\left(\frac{4}{n}\right)\left(\frac{3}{n}\right)\left(\frac{2}{n}\right)\boxed{\frac{1}{n}} \\
&\leq 1\cdot 1\cdot 1\cdot 1\cdot\ldots\cdot 1\cdot 1\cdot 1\cdot\boxed{\frac{1}{n}} \\
&= \boxed{\frac{1}{n}}
\end{aligned}$$

Since $\frac{1}{n}$ shoots to 0 as n marches to infinity, the smaller terms $\frac{n!}{n^n}$ must also approach 0. Question: Can you find a more helpful bound for the future? Can you bound $\frac{n!}{n^n}$ by $\frac{1}{n^2}$?

- To prove $\lim_{n\to\infty} \frac{4^n}{n!} = 0$, we work to find a bound on $\frac{4^n}{n!}$ for n large.

$$\begin{aligned}
\frac{4^n}{n!} &= \frac{4\cdot 4\cdot 4\cdot 4\cdot\ldots\cdot 4\cdot 4\cdot 4\cdot 4}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\cdot 4\cdot 3\cdot 2\cdot 1} \\
&= \boxed{\frac{4}{n}}\left(\frac{4}{n-1}\right)\left(\frac{4}{n-2}\right)\left(\frac{4}{n-3}\right)\cdots\left(\frac{4}{4}\right)\left(\frac{4}{3}\right)\left(\frac{4}{2}\right)\left(\frac{4}{1}\right) \\
&\leq \boxed{\frac{4}{n}}\cdot 1\cdot 1\cdot 1\cdot\ldots\cdot\left(\frac{4}{3}\right)\cdot\left(\frac{4}{2}\right)\cdot\left(\frac{4}{1}\right) \\
&= \frac{4^4}{6}\cdot\boxed{\frac{1}{n}}
\end{aligned}$$

Again, since $\frac{1}{n}$ shoots to 0 as n marches to infinity, the smaller terms $\frac{4^n}{n!}$ must also approach 0. Question: Can you find a more helpful bound for the future? Can you bound $\frac{4^n}{n!}$ by $c\left(\frac{4}{5}\right)^{n-4}$?

Some Natural Log Tips

- Fix $r > 0$, to show $\lim_{n\to\infty} \frac{\ln n}{n^r}^{\left(\frac{\infty}{\infty}\right)} = 0$ we consider the related function $f(x) = \frac{\ln x}{x^r}$ and apply L'H Rule.

  $$\lim_{x\to\infty} \frac{\ln x}{x^r}^{\left(\frac{\infty}{\infty}\right)} \stackrel{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{rx^{r-1}} = \lim_{x\to\infty} \frac{1}{rx^{r-1}x} = \lim_{x\to\infty} \frac{1}{rx^{r-1+1}} = \lim_{x\to\infty} \frac{1}{rx^r} = 0$$

  That is, $\ln n$ **grows slower than any positive power of n, as n goes to infinity.** This fact will be most helpful in the future, when we need to bound $\ln n$ by a power of n. For n large, we can use any of the following bounds, (whichever ends up being the most helpful power)

  $$\ln n \le n \qquad \ln n \le \sqrt{n} \qquad \ln n \le n^{\frac{1}{4}} \qquad \dots \qquad \textbf{for } n \textbf{ large}$$

  The first two are true for all $n \ge 1$. For instance, you might want to show that the sequence $\left\{\frac{\ln n}{n}\right\}$ converges to 0, using the Squeeze Theorem. (How else could you show it?) Since $\ln n \ge 1$ for $n > e$, we can't show that $\frac{\ln n}{n} \le \frac{1}{n}$, but we can show that $\frac{\ln n}{n} \le \frac{1}{\sqrt{n}}$. When that is true, we apply the Squeeze Theorem,

  $$0 \le \frac{\ln n}{n} \le \frac{1}{\sqrt{n}}$$
  $$\searrow \qquad \swarrow$$
  $$0$$

  Here $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$. As a result, the middle term is forced to approach 0. Now let's convince ourselves that $\frac{\ln n}{n} \le \frac{1}{\sqrt{n}}$. First note that from above, for n large, $\ln n \le \sqrt{n}$ which implies that $\frac{\ln n}{n} \le \frac{\sqrt{n}}{n}$, and in turn $\frac{\ln n}{n} \le \frac{1}{\sqrt{n}}$.

  Ideally, you are just grabbing a bound for $\ln n$ with enough powers of n, so that after dividing by some denominator containing powers of n, you are left with powers of n *still* in the denominator, and hence will shoot to 0, as n shoots to infinity.

  For example, to bound $\frac{\ln n}{n^3}$, that's easy, just bound $\ln n \le n$, and the quotient $\frac{\ln n}{n^3} \le \frac{n}{n^3} \le \frac{1}{n^2}$ and $\frac{1}{n^2}$ shoots to 0 as a helpful upper bound.

  Then to bound, say, $\frac{\ln n}{\sqrt{n}}$, we could use the bound $\ln n \le n^{\frac{1}{4}}$. Then, $\frac{\ln n}{\sqrt{n}} \le \frac{n^{\frac{1}{4}}}{\sqrt{n}} = \frac{1}{n^{\frac{1}{4}}}$.

  Similarly, you can bound $\frac{\ln n}{n^2} \le \frac{1}{n}$, but can you bound $\frac{\ln n}{n^2} \le \frac{1}{n^{\frac{3}{2}}}$? We will see how the second bound might be more helpful, when we study series in more detail.