Math 12 Helpful Sequence and Series Information

Please find below some definitions and tips for sequences and series. We will pay careful attention to *what* they are, *how* they are defined, *how* to decide convergence of each, as well as *how* they are related. We will also list a few classic examples of series. Please see the other class handout, for a list of convergence tests for series.

- **Sequences**

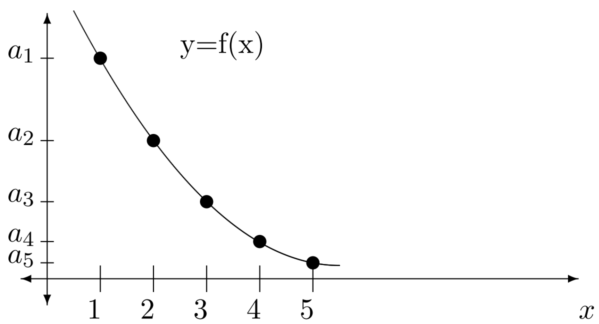
Definition: An infinite **sequence** of real numbers is an ordered, unending list of numbers.

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_{n-1},\ a_n,\ a_{n+1},\ \ldots$$

It can be difficult to represent an infinite list of numbers, so we abbreviate the sequence terms with a concise notation. Here n is an integer counter for the terms.

$$\{a_n\}_{n=1}^{\infty} \qquad \text{or} \qquad \{a_n\}_1^{\infty} \qquad \text{or} \qquad \{a_n\}_{n\geq 1} \qquad \text{or even} \quad \{a_n\} \text{ for short}$$

Think of the terms of a sequence as defined by a single, related function where the terms $a_n = f(n)$. Essentially, we have a formula for the terms of the sequence. The terms of the sequence can be represented as output plot points on the graph of $f(x)$, when you restrict the domain of f to a domain of positive integers n. Below $f(x)$ is drawn as the curve, and the dots are the sequence terms plotted along this curve. Since the sequence terms lie along the curve $f(x)$, it makes sense that the behavior of the sequence terms might be related to the behavior of the function.



Many questions arise:

- Does a list of numbers approach a fixed number or limit L?

- Not only does the list get close to a limit L, but does it remain close to L?

- If some limit L exists, how do we compute that number L?

We have several methods for **computing** limits of sequences.

- **Method 1:** $\varepsilon - N$ Proofs

  Definition: We say the **sequence** $\{a_n\}$ **converges** to a real number L, or has the limit L, and we write $\boxed{\lim_{n\to\infty} a_n = L}$ provided a_n can be made as close to L as we please, merely by choosing n to be sufficiently large. That is, given any $\varepsilon > 0$, there exists an integer N such that $|a_n - L| < \varepsilon$ for all $n \geq N$.

  Note: if the sequence $\{a_n\}$ does *not* converge, then we say it **diverges**.

  As with Math 11 $\varepsilon - \delta$ proofs, these proofs are challenging. The good news is that we have enough ammunition from Calculus 1 limits, and we can avoid dealing with these proofs.

- **Method 2:** Function Relationship

  Theorem: Given a sequence $\{a_n\}$, if there is related function $f(x)$ so that the terms of our sequence $a_n = f(n)$ and if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$

  Here we can use all of our previous limit techniques for functions. Please note that if you are going to apply L'H Rule, then technically you must really step aside and look at the related function f of x, since L'H Rule is not stated for terms of sequences.

  Caution: If $a_n = f(n)$ and $\lim_{x\to\infty} f(x)$ diverges, it does *not* necessarily mean that $\{a_n\}$ diverges. Can you think of a graph example? Try a function that oscillates.

Example: Does the sequence $\left\{\frac{(\ln n)^2}{n}\right\}_{n=1}^{\infty}$ converge?

To answer this question, we need to step aside to the related function $f(x) = \frac{(\ln x)^2}{x}$. Note

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x}^{\left(\frac{\infty}{\infty}\right)} \overset{L'H}{=} \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} = 2\lim_{x\to\infty} \frac{\ln x}{x}^{\frac{\infty}{\infty}} \overset{L'H}{=} 2\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Finally, since $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = 0$, then our theorem yields $\lim_{x\to\infty} \frac{(\ln n)^2}{n} = 0$, and therefore the original sequence converges to 0.

Example: Does the sequence $\left\{\left(\frac{n}{n+1}\right)^n\right\}_{n=1}^{\infty}$ converge?

To answer this question, we need the related function $f(x) = \left(\frac{x}{x+1}\right)^x$. Note

$$\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{x\,(1^\infty)} = \lim_{x\to\infty} e^{\ln\left(\frac{x}{x+1}\right)^x} = e^{\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)} = e^{\lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}}}$$

$$\overset{L'H}{=} e^{\lim_{x\to\infty} \frac{\left(\frac{x+1}{x}\right)\cdot\left(\frac{(x+1)(1)-x(1)}{(x+1)^2}\right)}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty} \frac{\left(\frac{x+1}{x}\right)\cdot\left(\frac{1}{(x+1)^2}\right)}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty} \frac{-x}{x+1}} = e^{-1} = \frac{1}{e}$$

Finally, since $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \frac{1}{e}$, then our theorem yields $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$, and therefore the original sequence converges to $\frac{1}{e}$.

- **Method 3:** Limit Laws for Sequences

  If $\{a_n\}$ and $\{b_n\}$ are both convergent sequences and c is a constant, then

  1. $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
  2. $\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
  3. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
  4. $\lim_{n\to\infty} c = c$
  5. $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
  6. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided the denominator limit is non-zero.
  7. $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ if $p > 0$ and $a_n > 0$

Example: The sequence $\left\{\frac{n^3}{7n^3+1}\right\}$ converges to $\frac{1}{7}$ since, using the Limit Laws,

$$\lim_{n\to\infty}\frac{n^3}{7n^3+1} = \lim_{n\to\infty}\frac{n^3}{7n^3+1}\cdot\frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3}\right)} = \lim_{n\to\infty}\frac{1}{7+\frac{1}{n^3}} \stackrel{L.L.}{=} \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 7 + \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} n^3}} = \frac{1}{7}$$

Sometimes it is difficult to convert your sequence terms to a helpful related function, but it is easier to find a useful bound or make a size argument. The following theorem is also known as the Squeeze Law, Sandwich Theorem, or even the Pinching Theorem.

- **Method 4:** Squeeze Theorem

  If $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences satisfying the following:

$$\left\{\begin{array}{c} \text{for all } \; n \geq N_0 \; \text{ we have } \; a_n \leq b_n \leq c_n \; \text{ for some } \; N_0 \\ \\ \text{and} \\ \\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \end{array}\right\} \quad \text{then} \quad \lim_{n\to\infty} b_n = L.$$

This fact makes sense. If the terms are eventually ordered, and the smallest and largest terms are approaching the same limit, then obviously the middle term, squished between those two terms, must approach the same limit. This rule is helpful when $L = 0$.

$$\begin{array}{ccccc} a_n & \leq & b_n & \leq & c_n \\ \searrow & & \downarrow & & \swarrow \text{ as } n \to \infty \\ & & L & & \end{array}$$

The Squeeze Law is good for terms with a bounded piece like $\sin n$, $\cos^2 n$ or even $\arctan n$. See other class handout about sequences, to study the bounded arguments for terms like $\frac{n!}{n^n}$.

Example: Does the sequence $\left\{\frac{\arctan n}{n^3}\right\}_{n=1}^{\infty}$ converge?

First note that, for each n, the terms $\frac{\arctan n}{n^3}$ are bounded both above and below. Each of those bounding terms shoot to 0 as n blows up to infinity. Therefore the terms bounded in the middle must also shoot to 0, by the Squeeze Law.

$$\begin{array}{ccccc} 0 & \leq & \frac{\arctan n}{n^3} & \leq & \frac{\pi}{2n^3} \\ \searrow & & \downarrow & & \swarrow \text{ as } n \to \infty \\ & & 0 & & \end{array}$$

- **Method 5:** Passing limits past continuous functions (to hit internal terms directly)

  Theorem: If $\lim_{n\to\infty} a_n = L$, and f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$.

  Example: Does $\left\{e^{\frac{1}{n^2}}\right\}_{n=1}^{\infty}$ converge? It does converge, to 1, since after passing the limit past the continuous exponential function, we have

$$\lim_{n\to\infty} e^{\frac{1}{n^2}} = e^{\left(\lim_{n\to\infty} \frac{1}{n^2}\right)} \overset{\text{L.L.}}{=} e^{\left(\frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} n^2}\right)} = e^0 = 1.$$

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Bounded Monotonic Sequence Property: This is used to prove several of our future convergence results/tests.

Theorem: Every bounded, monotonic sequence converges.

Caution: This fact does not tell you what the limit is, but it is less than the bound.

- **Series**

Definition: A **series** $\sum_{n=1}^{\infty} a_n$ is an infinite sum of numbers

$$a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n + a_{n+1} + \ldots$$

Here the counter index n is just a "dummy" variable that helps you keep track of the terms of the series. Notationally we have $\sum_{n=1}^{\infty} a_n = \sum_{j=1}^{\infty} a_j = \sum_{i=1}^{\infty} a_i = \sum_{k=1}^{\infty} a_k$. For each integer k, say, you get a term a_k, as the integer k ranges from 1 to ∞.

Many questions arise:

- Does it make sense to add up infinitely many numbers? How?

- Does that sum equal a finite number?

- If it exists, can we find the actual finite sum for a given series?

The third question turns out to be very difficult to answer for most series. To answer the first questions, we will use our traditional calculus approach of making a finite approximation to the item at hand, and then throw in a limit. To get a sense of this sum of infinitely many terms, let's examine a few finite sums of the first few terms of the series.

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

Each finite sum above is obviously some finite number. Note that these finite approximating sums create a list of numbers (a sequence). What happens to this list as you add on more and more terms (i.e. let the number of terms, n, march to infinity)?

Definition:

- Given a series $\sum_{n=1}^{\infty} a_n$, define the n^{th} **Partial Sum** as the sum of the first n terms,

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \\ &= \sum_{i=1}^{n} a_i \end{aligned}.$$

- If the sequence $\{S_n\}_{n\geq 1}$ of partial sums converges with $\lim_{n\to\infty} S_n = L$, we say the series $\sum_{n=1}^{\infty} a_n$ **converges** with **sum** L. We write $\sum_{n=1}^{\infty} a_n = L$. That is,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

- If the sequence $\{S_n\}_{n\geq 1}$ of partial sums diverges, we say the series $\sum_{n=1}^{\infty} a_n$ **diverges**. This could happen when $\lim_{n\to\infty} S_n = \pm\infty$ or Does Not Exist.

Example: The classic Telescoping Series (or collapsing series) is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. At first glance, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)} + \ldots$, and it is not obvious what this sums to. When you make a partial fractions decomposition of the series terms, everything changes.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

If we write out the n^{th} partial sum:

$$\begin{aligned} S_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

All middle terms cancel, and the original series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ equals $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$

Sometimes, analyzing the limit of the sequence of partial sums is difficult, so we will learn several convergence tests:

- n^{th} term Divergence test
- p-Test
- Geometric Series Test
- Integral Test
- (Direct) Comparison Test
- Limit Comparison Test
- Alternating Series Test
- Ratio Test
- Root Test
- Absolute Convergence Test

Please see the other class handout for the fine details of these convergence tests. We will reference a few of these special cases here.

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges, then the terms a_n must go to 0, that is, $\lim_{n\to\infty} a_n = 0$.

The contrapositive statement is also true, stating that *if* the terms do **not** go to zero, *then* there is **no** chance of that series converging. We give this test for divergence a name.

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n^{th} **term Divergence Test**: If $\lim_{n\to\infty} a_n \neq 0$ or Does Not Exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

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CAUTION: The converse statement is **not** true. Just because the terms a_n of a series go to 0 as $n \to \infty$ does **not** mean that the series converges. Take a classic example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. The terms approach 0 as n approaches infinity, but we can show, either using a size argument, or the integral test, that it is a divergent series.

If the terms of a series approach 0, all that means is "possible" convergence, and you have **more work** to do. You will press onward to use other available series convergence tests listed above.

We list a few of our favorite series.

Geometric series: Take $a \neq 0$. A geometric series is one of the following form, where each successive term is obtained from the previous term by multiplying by some number r, called the common ratio:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots ar^{n-1} + ar^n + \ldots$$

A geometric series converges if $|r| < 1$ and diverges otherwise.

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} = \dfrac{a}{1-r} & \text{if} \quad |r| < 1 \\ \text{diverges} & \text{if} \quad |r| \geq 1 \end{cases}$$

Think about the common ratio r being raised to a large power. If r is tiny then r^{n-1} will shrink to 0 when n is large. The best advice is to write out the first few terms, pick off the first term a, pick off the multiplying ratio r, check if $|r| < 1$, if it is, find the sum $\dfrac{a}{1-r}$.

***p*-series:** $$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots + \frac{1}{n^p} + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if} \quad p > 1 \\ \text{diverges} & \text{if} \quad p \leq 1 \end{cases}$$

We proved this p-Test result using the convergence results we had from Section 8.8 regarding improper p-integrals, along with the Integral Test. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent exactly when $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ is. Recall,

$$\int_1^{\infty} \frac{1}{x^p}\,dx \begin{cases} \text{converges} & \text{if} \quad p > 1 \qquad \text{(what does it converge to?)} \\ \text{diverges} & \text{if} \quad p \leq 1 \end{cases}$$

Note: It is important to solidify the difference between a Geometric series and a p-series. You want to immediately recognize these two series. For Geometric series, the variable n is in the exponent, but for p-series, the variable n is in the base. Please learn the convergence conditions carefully, since they are easy to mix up. The inequalities $<$ vs. $\leq$ and $>$ vs. $\geq$ are different for each.

Keep in mind the difference between a sequence and a series. A sequence is a list of numbers, and a series is a sum of numbers. For each series $\sum_{n=1}^{\infty} a_n$ there are two related, but different, sequences:

- the sequence of the terms $\{a_n\}$

and

- the sequence of partial sums $\{S_n\}$

.

The question about convergence of a series is reduced to the question of convergence of a sequence. For a series $\sum_{n=1}^{\infty} a_n$ to converge, we need two things: the terms of the sequence must approach 0 as n gets large, and the sequence of partial sums must converge.

- $\lim_{n\to\infty} a_n = 0$

and

- $\lim_{n\to\infty} S_n = L$

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Example: Discuss whether the sequence $\left\{\left(\frac{n}{n+1}\right)^n\right\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$ converge.

We showed earlier in this paper that $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$. Therefore

$$\boxed{\text{the sequence } \left\{\left(\frac{n}{n+1}\right)^n\right\}_{n=1}^{\infty} \textbf{ converges} \text{ to } \frac{1}{e}}.$$

When we consider the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$, we recognize that the terms are **not** approaching 0, they are approaching $\frac{1}{e}$. Therefore,

$$\boxed{\text{the series } \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n \textbf{ diverges}}$$

by the n^{th} Term Divergence Test. The terms of the series **do** converge to $\frac{1}{e}$, but because that limit is not 0, we can **immediately** conclude something about the divergence of the series.

Example: Discuss whether the sequence $\left\{\dfrac{3n^2+7}{5n^2+1}\right\}_{n=1}^{\infty}$ and the series $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2+7}{5n^2+1}$ converge.

We can easily show that $\displaystyle\lim_{n\to\infty}\frac{3n^2+7}{5n^2+1}=\frac{3}{5}$ (how?). Therefore

$$\boxed{\text{the sequence } \left\{\frac{3n^2+7}{5n^2+1}\right\}_{n=1}^{\infty} \textbf{ converges} \text{ to } \frac{3}{5}}.$$

When we consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2+7}{5n^2+1}$, we recognize that the terms are **not** approaching 0, they are approaching $\frac{3}{5}$. Therefore, by the n^{th} term divergence test,

$$\boxed{\text{the series } \sum_{n=1}^{\infty}\frac{3n^2+7}{5n^2+1} \textbf{ diverges}}.$$

Bounds: When working with series, it is sometimes helpful to compare our given series with a another series whose convergence is easier to conclude. We will sometimes need certain bounds on functions. We mention just a few here. **As n gets large,**

- $1\le \ln n\le n \quad\Longrightarrow\quad \dfrac{1}{n}\le\dfrac{\ln n}{n}\le 1$
- $n\le n! \quad\Longrightarrow\quad \dfrac{1}{n!}\le\dfrac{1}{n}$
- $n\le 2^n$ or in general $n\le a^n$ for $a>1$
- $2^n\le n! \quad\Longrightarrow\quad \dfrac{1}{n!}\le\dfrac{1}{2^n}$
- $\ln n\le n^r$ for some $r>0$

Example: Discuss whether the sequence $\left\{\dfrac{1}{n!}\right\}_{n=1}^{\infty}$ and the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ converge.

Hint: Of the two bounds listed above for $\frac{1}{n!}$, one of them is helpful enough to answer the convergence question for the sequence (Squeeze Law?), and the other one is more helpful to answer the convergence question for the series. Why is knowing that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ seems smaller than $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ not enough to declare convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$? Why is knowing that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ seems smaller than $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ good enough to declare convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$? You had this problem on your homework.