Review Packet for Exam #2-Second Part

Math 12-D. Benedetto

Series: Find the **sum** for each of the following series (all of which converge):

Sum of 2 Convergent Geometric Series

96. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{6^n} = \sum_{n=1}^{\infty} \frac{2^n}{6^n} + \sum_{n=1}^{\infty} \frac{3^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$ ✓

① $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} + \frac{1}{3^2} + \cdots = \frac{a}{1-r} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{2}$

$a = 1/3 \quad r = 1/3$

② $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{2^2} + \cdots = \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$

$a = 1/2 \quad r = 1/2$

97. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n} - \frac{1}{7^n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{7}\right)^n = \frac{4}{3} - \frac{7}{6} = \frac{8}{6} - \frac{7}{6} = \boxed{\frac{1}{6}}$ ✓

① $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots = \frac{a}{1-r} = \frac{1}{1-1/4} = \frac{1}{3/4} = \frac{4}{3}$

$a = 1 \quad r = 1/4$

Difference of 2 Convergent Geometric Series

② $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{7}\right)^n = 1 + 1/7 + \frac{1}{7^2} + \cdots = \frac{a}{1-r} = \frac{1}{1-1/7} = \frac{1}{6/7} = 7/6$

$a = 1 \quad r = 1/7$

98. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{n-1}}{3^{n+1}} = \frac{1}{9} - \frac{2}{27} + \cdots = \frac{a}{1-r} = \frac{1/9}{1-(-2/3)} = \frac{\frac{1}{9}}{5/3} = \frac{1}{9} \cdot \frac{3}{5} = \boxed{\frac{1}{15}}$ ✓

$a = 1/9 \quad r = -\frac{2}{3}$

Convergent Geometric Series

$|r| = 2/3 < 1$

99. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{4n-1}} = \frac{3^3}{2^3} + \frac{3^4}{2^7} + \frac{3^5}{2^{11}} + \cdots = \frac{a}{1-r} = \frac{27/8}{1-3/16} = \frac{27/8}{13/16} = \frac{27}{8} \cdot \frac{16}{13} = \boxed{\frac{54}{13}}$ ✓

$a = \frac{27}{8} \quad r = \frac{3}{2^4} = 3/16$

Convergent Geometric Series

$|r| = 3/16 < 1$

Telescoping Series ✓

100. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \cancelto{0}{\frac{1}{\sqrt{n+1}}} = \boxed{1}$

nth Partial Sum: $S_n = 1 - \cancel{1/\sqrt{2}} + \cancel{1/\sqrt{2}} - \cancel{1/\sqrt{3}} + \cancel{1/\sqrt{3}} - \cdots + \cancel{\frac{1}{\sqrt{n-1}}} - \cancel{\frac{1}{\sqrt{n}}} + \cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{n+1}}$

101. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3\cdot 2^n} = \frac{1}{6} - \frac{1}{12} + \frac{1}{24} + \cdots = \frac{a}{1-r} = \frac{1/6}{1-(-1/2)} = \frac{1/6}{3/2} = \frac{1}{6}\cdot\frac{2}{3} = \boxed{\frac{1}{9}}$ ✓

$a = 1/6 \quad r = -1/2$

Convergent Geometric Series

$|r| = 1/2 < 1$

102. $\displaystyle\sum_{n=1}^{\infty} e^{\frac{1}{n}} - e^{\frac{1}{n+1}}$ telescoping series $= e^1 - e^{\frac{1}{2}} + e^{1/2} - e^{1/3} + e^{1/3} - e^{1/4} + \cdots = \lim_{n\to\infty} S_n = \lim_{n\to\infty} e - e^{1/n+1} = \boxed{e-1}$ ✓

n^{th} Partial Sum $S_n = e^1 - \cancel{e^{1/2}} + \cancel{e^{1/2}} - \cancel{e^{1/3}} + \cdots + \cancel{e^{1/n-1}} - \cancel{e^{1/n}} + \cancel{e^{1/n}} - e^{1/n+1} = e^1 - e^{\frac{1}{n+1}}$

103. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{3^{2n-1}} = \frac{4}{3} + \frac{4^2}{3^3} + \frac{4^3}{3^5} + \cdots = \frac{a}{1-r} = \frac{4/3}{1-4/9} = \frac{4/3}{5/9} = 4/3 \cdot 9/5 = \boxed{\frac{12}{5}}$ ✓

$a = 4/3 \quad r = \frac{4}{3^2} = 4/9$

or $= \displaystyle\sum \frac{4^n}{3^{2n} 3^{-1}} = \sum_{n=1}^{\infty} \frac{3\cdot 4^n}{9^n}$

$= \underbrace{3\left(\frac{4}{9}\right)}_{4/3} + 3\cdot\left(\frac{4}{9}\right)^2$

Convergent Geometric Series

$|r| = 4/9 < 1$

104. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \cancelto{0}{\frac{1}{n+1}} = \boxed{1}$ ✓

telescoping series (Partial Fraction)

n^{th} Partial Sum $S_n = \displaystyle\sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \cdots + \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$

105. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{9^{n-1}} = \frac{-4}{1} + \frac{4^2}{9} - \frac{4^3}{9^2} + \cdots = \frac{a}{1-r} = \frac{-4}{1-(-4/9)} = \frac{-4}{13/9} = -4\cdot 9/13 = \boxed{\frac{-36}{13}}$ ✓

$a = -4 \quad r = \frac{-4}{9}$

Convergent Geometric Series

$|r| = 4/9 < 1$

106. $\sum_{n=1}^{\infty} 2^{-2n} = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \sum_{n=1}^{\infty} \frac{1}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{a}{1-r} = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{4}\cdot\frac{4}{3} = \boxed{\frac{1}{3}}$ ✓

Convergent Geometric Series

$a = 1/4 \quad |r| = 1/4 < 1$

More Series: Determine whether each of the following series **converge** or **diverge**. Name any convergence test(s) you use, and justify that it's legal to use them:

107. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$ Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}(n+1)}{2^{n+1}}}{\frac{(-1)^n n}{2^n}}\right| = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \overbrace{\frac{n+1}{n}}^{1} \cdot \frac{2^n}{\underbrace{2^{n+1}}_{2^n \cdot 2}} = \frac{1}{2} < 1$$

$\Rightarrow$ Absolute Convergence By Ratio Test

$\Rightarrow$ original series $\boxed{\text{convergent}}$ ✓

108. $\sum_{n=1}^{\infty} \underbrace{\frac{2n + \ln n}{n + 2010}}_{a_n}$ $\boxed{\text{Diverges}}$ ✓ by n^{th} term Divergence Test

Method 1:

$$\lim_{x\to\infty} \frac{2x + \ln x}{x + 2010} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{2 + 1/x}{1} = 2 \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n + \ln n}{n + 2010} = 2 \neq 0$$

Method 2:

$$\lim_{n\to\infty} \frac{2n + \ln n\,(1/n)}{n + 2010\,(1/n)} = \lim_{n\to\infty} \frac{2 + \frac{\ln n}{n}}{1 + \frac{2010}{n}} = 2 \neq 0$$

b/c $\lim_{x\to\infty} \frac{\ln x}{x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{(1/x)}{1} = 0$

either way, terms of Series do <u>not</u> go to zero.

109. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ $\boxed{\text{Diverges}}$ ✓ by n^{th} term Divergence Test

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n^2} = \infty \neq 0$$

the terms of the series do <u>not</u> go to zero.

110. $\sum_{n=1}^{\infty} \frac{n}{\underbrace{(n+1)^2 - n}_{\underbrace{n^2+2n+1-n}_{n^2+n+1}}} \approx \sum \frac{1}{n}$ Try Limit Comparison Test

$$\lim_{n\to\infty} \frac{\frac{n}{(n+1)^2 - n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2 - n} = 1 \quad \text{finite, non-zero}$$

$\Rightarrow$ Both Series share same behavior

Since $\sum \frac{1}{n}$ diverges, then original $\sum \frac{n}{(n+1)^2 - n}$ $\boxed{\text{diverges}}$ ✓ by LCT.

• Try looking at Abs. Series $\sum \frac{n}{n^2+1} \approx \sum \frac{1}{n}$ it will diverge by L.C.T
so do not have absolute converge[nce]
to ensure convergence, left wi[th] A.S.T

111. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$

• Ratio Test not helpful here (will give $L=1$)

• Try Alternating Series Test

① $b_n = \frac{n}{n^2+1} > 0$

② $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{n}{n^2+1} = 0$

③ $b_{n+1} < b_n$ terms decreasing.

$\Rightarrow$ $\boxed{\text{converges by AST}}$ ✓

$f(x) = \frac{x}{x^2+1}$

$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$

$= \frac{-x^2+1}{\underbrace{(x^2+1)^2}_{+}}$ <0 when $-x^2+1<0$ or $x>1$ o.k here

112. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n n^2}{n!}$ Ratio Test

$$L = \lim_{n\to\infty}\frac{\frac{2^{n+1}(n+1)^2}{(n+1)!}}{\frac{2^n n^2}{n!}} = \lim_{n\to\infty}\frac{2^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{2^n n^2} = \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{n!}{(n+1)!}\cdot\frac{(n+1)^2}{n^2}$$

$$= \lim_{n\to\infty}\overbrace{\frac{2}{n+1}}^{\to 0}\cdot\overbrace{\left(\frac{n+1}{n}\right)^2}^{\to 1} = 0 < 1 \Rightarrow \boxed{\text{converges}} \text{ by Ratio Test ✓}$$

113. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ can use comparison if notice $\ln n \le \sqrt{n}$

Since $\frac{\ln n}{n^2} \le \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ and $\sum \frac{1}{n^{3/2}}$ is convergent ($p=3/2$) series $\Rightarrow \sum\frac{\ln n}{n^2}$ converge[s] by C.[T.]

or try Integral Test

$$\int_1^{\infty}\frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \ln x\,(x^{-2})\,dx = \lim_{t\to\infty} \ln x\left(-\frac{1}{x}\right)\Big|_1^t - \int_1^t -\frac{1}{x}\cdot\frac{1}{x}\,dx = \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_1^t + \frac{x^{-1}}{-1}\Big|_1^t$$

$u = \ln x \quad dv = x^{-2}dx$

$du = \frac{1}{x}dx \quad v = \frac{x^{-1}}{-1}$

$$= \lim_{t\to\infty} -\overbrace{\frac{\ln t}{t}}^{\to 0 \text{ by L'H}} - \left(-\frac{\ln 1}{1}\right) - \overbrace{\frac{1}{t}}^{\to 0} - \left(-\frac{1}{1}\right) = \boxed{1} \text{ finite ✓}$$

$\Rightarrow \sum\frac{\ln n}{n^2}$ $\boxed{\text{converges}}$ by Integral Test

[No]te:
$f(x) = \frac{\ln x}{x^2}$
positive, continuous here
decreasing
$f'(x) = \frac{x^2\cdot\frac{1}{x} - \ln x\cdot 2x}{x^4}$
$= \frac{x(1-2\ln x)}{x^4}$
$= \frac{1-2\ln x}{x^3} < 0$
when $1 < 2\ln x$
$\Rightarrow \ln x > \frac{1}{2}$
as soon as $x > e^{1/2}$

114. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{2n^2\sqrt{n}+9} \approx \sum\frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty}\frac{\frac{n^2+1}{2n^{5/2}+9}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{n^{5/2}+\sqrt{n}}{2n^{5/2}+9} = \boxed{\frac{1}{2}}$$

non-zero Finite $\Rightarrow$ Both have same behavior

Since $\sum\frac{1}{\sqrt{n}}$ diverges ($p=1/2<1$), then the original series also $\boxed{\text{diverges}}$ by L.C.T. ✓

115. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+3}{4n^2-2} \approx \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$

$$\lim_{n\to\infty}\frac{\frac{\sqrt{n}+3}{4n^2-2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\frac{n^2+3n^{3/2}}{4n^2-2} = \boxed{1/4}$$

non-zero Finite $\Rightarrow$ Both have same behavior

Since $\sum\frac{1}{n^{3/2}}$ converges ($p=3/2>1$), then the original series also $\boxed{\text{converges}}$ ✓ by L.C.T.

116. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{19}+40n^6+4n^3+19}{4+17n^5+n^{20}} \approx \sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{n^{19}+40n^6+4n^3+19}{4+17n^5+n^{20}}}{1/n} = \lim_{n\to\infty} \frac{n^{20}+40n^7+4n^4+19n}{4+17n^5+n^{20}} = \boxed{1}$$

Non-zero Finite $\Rightarrow$ both have same behavior

Since $\sum \frac{1}{n}$ diverges, then the original serie also $\boxed{\text{diverges}}$ by L.C.T. ✓

117. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n(\sqrt{n}+1)}$ look at $\sum \frac{|\sin n|}{n\sqrt{n}+n}$ absolute series.

Since $\frac{|\sin n|}{n^{3/2}+n} \le \frac{1}{n^{3/2}+n} \le \frac{1}{n^{3/2}}$ and $\sum \frac{1}{n^{3/2}}$ is a convergent p-series ($p=3/2>1$)

then the (smaller) series $\sum \frac{|\sin n|}{n^{3/2}+n}$ converges by CT

Then $\sum \frac{\sin n}{n^{3/2}+n}$ is absolutely convergent and therefore $\boxed{\text{convergent.}}$ ✓

118. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^n n!}$ Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{2^{n+1}(n+1)!}}{\frac{n^n}{2^n n!}} \right| = \lim_{n\to\infty} \frac{\overbrace{(n+1)^{n+1}}^{(n+1)(n+1)^n}}{2^{n+1}(n+1)!} \cdot \frac{2^n n!}{n^n} = \lim_{n\to\infty} \cancel{(n+1)} \underbrace{\left(\frac{n+1}{n}\right)^n}_{\to e} \cdot \frac{2^n}{\underbrace{2^{n+1}}_{2^n\cdot 2}} \cdot \frac{\cancel{n!}}{\underbrace{(n+1)!}_{\cancel{(n+1)}\,\cancel{n!}}} = \frac{e}{2} > 1 \Rightarrow \boxed{\text{Diverges}} \text{ by Ratio Test}$$ ✓

119. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\ln 2)^n}$ Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)(\ln 2)^{n+1}}}{\frac{1}{n(\ln 2)^n}} \right| = \lim_{n\to\infty} \frac{n(\ln 2)^n}{(n+1)(\ln 2)^{n+1}} = \lim_{n\to\infty} \underbrace{\frac{n}{n+1}}_{\to 1} \frac{(\ln 2)^n}{(\ln 2)^{n+1}} = \frac{1}{\ln 2} > 1 \text{ since } \ln 2 < 1$$

(we know $\ln e = 1$ and $2 < e$ and $\ln$ increasin

$\Rightarrow \boxed{\text{Diverges}}$ by Ratio Test ✓

120. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ Recall $\ln n \le \sqrt{n} \Rightarrow \frac{1}{\ln n} \ge \frac{1}{\sqrt{n}} \Rightarrow \frac{1}{(\ln n)^2} \ge \frac{1}{n}$

Since $\frac{1}{(\ln n)^2} \ge \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ($p=1$) $\Rightarrow$ original $\sum \frac{1}{(\ln n)^2}$ also $\boxed{\text{diverges}}$ ✓ by C.T.

121. $\sum_{n=1}^{\infty} \frac{\ln n}{e^n}$ Ratio Test

$L = \lim_{n\to\infty} \left| \frac{\frac{\ln(n+1)}{e^{n+1}}}{\frac{\ln n}{e^n}} \right| = \lim_{n\to\infty} \frac{\cancel{\ln(n+1)}^{1}}{\cancel{\ln n}} \cdot \frac{e^n}{e^{n+1}} = \frac{1}{e} < 1 \Rightarrow$ (absolutely) $\boxed{\text{convergent}}$ by Ratio Test ✓

$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$

needs to start at $n=2$

(Ratio test Inconclusive)

$\lim_{n\to\infty} \frac{\frac{1}{(n+1)\ln(n+1)}}{\frac{1}{n\ln n}} = \lim_{n\to\infty} \frac{\cancel{n}^1}{\cancel{n+1}} \frac{\cancel{\ln n}^1}{\cancel{\ln(n+1)}} = 1$

122. $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ Integral Test $f(x) = \frac{1}{x\ln x}$ positive, cont. decreasing because $f'(x) = \frac{-1}{(x\ln x)^2}\left[x \cdot \frac{1}{x} + \ln x(1)\right] < 0$

$\int_2^\infty \frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_{\ln 2}^{\ln t} \frac{1}{u}du = \lim_{t\to\infty} \ln|u| \Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} \cancel{\ln|\ln t|}^{\infty} - \ln|\ln 2| = \infty$

$u = \ln x \quad x=2 \Rightarrow \ln 2$

$du = \frac{1}{x}dx \quad x=t \Rightarrow u = \ln t$

Diverges

$\Rightarrow$ original series $\boxed{\text{Diverges}}$ ✓ by Integral Test

123. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3n+2}$ $\lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3} \neq 0$ so $\lim_{n\to\infty}(-1)^n \frac{n}{3n+2}$ D.N.E.

$\Rightarrow$ original series $\boxed{\text{diverges}}$ ✓ by nth term Divergence Test

124. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ Ratio Test

$L = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \cancel{\frac{3}{n+1}}^{0} = 0 < 1 \Rightarrow \boxed{\text{Converges}}$ ✓ (absolutely) by Ratio Test

125. $\sum_{n=1}^{\infty} n e^{-n^2} = \sum \frac{n}{e^{n^2}}$ Try Ratio test

$L = \lim_{n\to\infty} \left| \frac{\frac{n+1}{e^{(n+1)^2}}}{\frac{n}{e^{n^2}}} \right| = \lim_{n\to\infty} \frac{\cancel{n+1}^{1}}{n} \frac{e^{n^2}}{\underbrace{e^{n^2+2n+1}}_{e^{n^2}e^{2n+1}}} = \lim_{n\to\infty} \cancel{\frac{1}{e^{2n+1}}}^{0} = 0 < 1 \Rightarrow \boxed{\text{Converges}}$ ✓ (absolutely) by Ratio Test

positive, cont.

$f(x) = \frac{x}{e^{x^2}}$ decreasing

$f'(x) = \frac{e^{x^2} - x e^{x^2} \cdot 2x}{(e^{x^2})^2} = \frac{e^{x^2}(1-2x^2)}{(e^{x^2})^2} = \frac{1-2x^2}{e^{x^2}} < 0$ when $x^2 > \frac{1}{2}$

$\boxed{\text{OR}}$ try Integral Test

$\int_1^\infty x e^{-x^2}dx = \lim_{t\to\infty}\int_1^t x e^{-x^2}dx = \lim_{t\to\infty} -\frac{1}{2}\int_{-1}^{-t^2} e^u du = \lim_{t\to\infty} -\frac{1}{2}e^u \Big|_{-1}^{-t^2} = \lim_{t\to\infty} \cancel{-\frac{1}{2}e^{-t^2}}^{0} - \left(-\frac{1}{2}e^{-1}\right) = \frac{1}{2e}$ finite

$u = -x^2 \quad x=1 \Rightarrow u=-1$

$du = -2x\,dx \quad x=t \Rightarrow u = -t^2$

$\Rightarrow$ series $\boxed{\text{converges}}$ ✓ by Integral Test

126. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{\ln n}$ $\boxed{\text{Diverges}}$ by n^{th} term Divergence Test.

$$\lim_{x\to\infty} \frac{x}{\ln x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1}{(1/x)} = \lim_{x\to\infty} x = \infty \Rightarrow \lim_{n\to\infty} \frac{n}{\ln n} \neq 0 \Rightarrow \lim_{n\to\infty} (-1)^{n+1} \frac{n}{\ln n} \text{ D.N.E.}$$

127. $\sum_{n=1}^{\infty} \frac{n!}{10^{4n}}$ Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{10^{4(n+1)}}}{\frac{n!}{10^{4n}}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{\underbrace{10^{4n+4}}_{10^{4n}\cdot 10^4}} \cdot \frac{10^{4n}}{n!} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{n!}} \cdot \frac{\cancel{10^{4n}}}{\cancel{10^{4n}}\cdot 10^4} = \lim_{n\to\infty} \frac{n+1}{10^4} = \infty > 1$$

$\Rightarrow \boxed{\text{Diverges}}$ by Ratio Tes

128. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{7}{8}}}$ $\boxed{\text{Divergent}}$ p-series $p = 7/8 < 1$.

129. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \approx \sum \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}^{\,1} = 1$$

non-zero finite $\Rightarrow$ both have same behavior

Since $\sum \frac{1}{\sqrt{n}}$ is divergent p-series $p = \frac{1}{2} < 1$

$\Rightarrow \sum \frac{1}{\sqrt{n+1}}$ also $\boxed{\text{divergent}}$ by L.C.T.

130. $\sum_{n=1}^{\infty} e^{-2n} = \sum \left(\frac{1}{e^2}\right)^n$ Geometric $|r| = 1/e^2 < 1 \Rightarrow \boxed{\text{convergent}}$.

OR $= \sum \frac{1}{e^{2n}}$ and do Ratio Test

$$\lim_{n\to\infty} \frac{\frac{1}{e^{2(n+1)}}}{\frac{1}{e^{2n}}} = \lim_{n\to\infty} \frac{e^{2n}}{e^{2n+2}} = \frac{1}{e^2} < 1 \quad \text{convergent}$$

131. $\sum_{n=1}^{\infty} \frac{1+3n^3}{n^5} \approx \sum \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{\frac{1+3n^3}{n^5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2+3n^5}{n^5} = 3$ non-zero finite $\Rightarrow$ both have same behavior.

Since $\sum \frac{1}{n^2}$ conv. p-series $p=2>1$

then original series also $\boxed{\text{converges}}$ ✓ by L.C.T.

132. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(\ln n)^2}$ Note: $\ln n \le \sqrt{n} \Rightarrow \frac{1}{\ln n} \ge \frac{1}{\sqrt{n}} \Rightarrow \frac{1}{(\ln n)^2} \ge \frac{1}{n}$

☹ need $n=2$ here

Since $\frac{\sqrt{n}}{(\ln n)^2} \ge \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ and $\sum \frac{1}{\sqrt{n}}$ Diverges ($p=\frac{1}{2}<1$) then the original (larger) series $\boxed{\text{diverges}}$ ✓ by C.T.

or Diverges by nth term test $\lim_{n\to\infty} \frac{\sqrt{n}}{(\ln n)^2} \overset{=\infty}{\ne} 0$ since

$\lim_{x\to\infty} \frac{\sqrt{x}}{(\ln x)^2} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{2\ln x \cdot \frac{1}{x}} = \lim_{x\to\infty} \frac{x}{4\sqrt{x}\ln x} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln x} \overset{\infty/\infty}{}$

$\underset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\left(\frac{4}{x}\right)} = \lim_{x\to\infty} \frac{x}{8\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{8} = \infty$

133. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ Ratio Test

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{2}{n+1} \overset{0}{=} 0 < 1$

$\Rightarrow \boxed{\text{converges}}$ ✓ by Ratio Test

needs to be $n=2$

134. $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^7}$ Integral Test $f(x) = \frac{1}{x(\ln x)^7}$ cont./pos./decreasing $\Rightarrow f'(x) = \frac{-1}{[x(\ln x)^7]^2}\left[x \cdot 7(\ln x)^6 \frac{1}{x} + (\ln x)^7\right] = \underbrace{\frac{-1}{[x(\ln x)^7]^2}(\ln x)^6[7+\ln x]}_{<0}$

$\int_2^{\infty} \frac{1}{x(\ln x)^7}dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^7}dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^7}du = \lim_{t\to\infty} \frac{u^{-6}}{-6}\Big|_{\ln 2}^{\ln t}$

$-\left(\frac{-1}{6(\ln 2)^6}\right) = \frac{1}{6(\ln 2)^6}$ Finite $\Rightarrow$ original series ✓ $\boxed{\text{converges}}$ by Integral Test

135

$\frac{\pi/2}{1+n^2} \le \frac{\pi/2}{n^2}$ and $\frac{\pi}{2}\sum \frac{1}{n^2}$ converges (constant multiple of convergent is convergent; conv. $p=2>1$)

) series $\boxed{\text{converges}}$ ✓ by C.T.

or

$f(x) = \frac{\arctan x}{1+x^2}$

$f'(x) = \frac{(1+x^2)\left(\frac{1}{1+x^2}\right) - \arctan x \cdot 2x}{(1+x^2)^2}$

$= \frac{1-2x\arctan x}{(1+x^2)^2} < 0$ when $1 < 2x\arctan x$

this is obviously more work

$\int_{\pi/4}^{\arctan t} u\,du = \lim_{t\to\infty} \frac{u^2}{2}\Big|_{\pi/4}^{\arctan t} = \lim_{t\to\infty} \frac{(\arctan t)^2}{2} - \frac{(\pi/4)^2}{2}$

$u = \arctan 1 = \pi/4$

$u = \arctan t$

$= \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right] = \frac{1}{2}\left[\frac{3\pi^2}{16}\right] = \boxed{\frac{3\pi^2}{32}}$ ✓

Finite

136. $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2}$ Since $\dfrac{2+\sin n}{n^2} \le \dfrac{3}{n^2}$ and $\underbrace{3\overbrace{\sum \frac{1}{n^2}}^{\text{p-series conv. } p=2>1}}_{\text{constant multiple of convergent is convergent}}$ converges

then the original (smaller) series $\boxed{\text{converges}}$ by C.T.

137. $\displaystyle\sum_{n=1}^{\infty} \frac{n^7}{e^n}$ Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^7}{e^{n+1}}}{\frac{n^7}{e^n}}\right| = \lim_{n\to\infty} \frac{(n+1)^7}{e^{n+1}} \cdot \frac{e^n}{n^7} = \lim_{n\to\infty} \cancelto{1}{\left(\frac{n+1}{n}\right)^7} \cdot \frac{e^n}{e^{n+1}} = \frac{1}{e} < 1$$

$\Rightarrow$ $\boxed{\text{converges}}$ by Ratio Test

$n! = n(n-1)\cdots 3\cdot 2\cdot 1$, $3^n = 3\cdot 3\cdots 3\cdot 3\cdot 3$; $\ge 1 \ge$, $= 1$; $\ge \dfrac{2n}{27}$

138. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$ Ratio Test or Diverges by n^{th} Term Div. Test since $\displaystyle\lim_{n\to\infty} \frac{n!}{3^n} \ge \lim_{n\to\infty} \frac{2n}{27} = \infty$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}}\right| = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{3} = \infty > 1$$

$\Rightarrow$ $\boxed{\text{Diverges}}$ by Ratio Test

139. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{5n^3+3n^2} \approx \sum \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{2n+5}{5n^3+3n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^3+5n^2}{5n^3+3n^2} = \frac{2}{5}$$ non-zero Finite $\Rightarrow$ both series share same behavior

Since $\sum \frac{1}{n^2}$ conv. p-series $p=2>1$, then

the original series is also $\boxed{\text{conv.}}$ by L.C.T.

140. $\displaystyle\sum_{n=1}^{\infty} \underbrace{(e^n+n)^{\frac{1}{n}}}_{a_n}$ $\boxed{\text{Diverges}}$ by n^{th} Term Div. Test.

$\lim_{n\to\infty} a_n \neq 0$ since $$\lim_{x\to\infty} \underbrace{(e^x+x)^{1/x}}_{\infty^0} = e^{\lim_{x\to\infty} \frac{1}{x}\ln(e^x+x)\ \frac{\infty}{\infty}} \underset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{e^x+x}(e^x+1)}{1}\ \frac{\infty}{\infty}} \underset{L'H}{=} e^{\lim_{x\to\infty} \frac{e^x}{e^x+1}\ \frac{\infty}{\infty}} \underset{L'H}{=} e^{\lim_{x\to\infty} \cancelto{1}{\frac{e^x}{e^x}}} = e \neq 0$$

$$\frac{n^n}{n!} = \frac{n \cdot n \cdots n \cdot n \cdot n}{n(n-1)\cdots 3 \cdot 2 \cdot 1} \geq n$$

Div. by n^{th} Term Divergence Test

141. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $\quad \lim_{n\to\infty} \frac{n^n}{n!} \geq \lim_{n\to\infty} n = \infty$

Or use Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} \right| = \lim_{n\to\infty} \frac{\overbrace{(n+1)^{n+1}}^{(n+1)(n+1)^n}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \cancel{(n+1)} \left(\frac{n+1}{n}\right)^n \cdot \frac{n!}{\cancel{(n+1)!}} = e > 1 \Rightarrow \boxed{\text{Div.}} \text{ by Ratio Test}$$

142. $\sum_{n=1}^{\infty} n^{\frac{1}{n}}$ $\quad \boxed{\text{Div.}}$ by n^{th} term Divergence Test $\quad$ Recall $\lim_{n\to\infty} n^{1/n} = 1$

$$\lim_{n\to\infty} n^{1/n} = 1 \neq 0 \text{ since } \lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty} \frac{1}{x}\ln x} \overset{L'H}{=} e^{\lim_{x\to\infty} \frac{1/x}{1}} = e^0 = \boxed{1} \neq 0$$

terms not approaching 0.

143. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\quad$ Note that $\ln n \geq 1$ (for $n \geq 3$) then $\frac{\ln n}{n} \geq \frac{1}{n}$.

Since $\sum \frac{1}{n}$ is a divergent ($p=1$) series

then the original (larger) series $\boxed{\text{diverges}}$ by C.T.

$f(x) = \frac{\ln x}{x}$ pos./cont/decreasing

$f'(x) = \frac{x \cdot 1/x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$ when $1 < \ln x$ or $x > e$

or integral test

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big|_1^t = \lim_{t\to\infty} \frac{(\ln t)^2}{2} - \frac{(\ln 1)^2}{2} = \infty \Rightarrow \text{series diverges by Integral Test}$$

TYPO! 144. $\cancel{\sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^2 + 3}}$ Not Solvable in Math 12

Instead look at $\sum \frac{n \sin^2 n}{n^3 + 3}$. Since $\frac{n \sin^2 n}{n^3+3} \leq \frac{n}{n^3+3} \leq \frac{n}{n^3} = \frac{1}{n^2}$

and since $\sum \frac{1}{n^2}$ ($p=2$) is convergent p-series.

the the original (smaller) series $\boxed{\text{converges}}$ by CT

145. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ $\quad$ Note $\sum \frac{1}{n^2}$ is convergent p-series $p=2$.

$\Rightarrow \sum \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent

and therefore $\boxed{\text{convergent}}$

$\boxed{\text{Or}}$ can use AST

① $b_n = \frac{1}{n^2} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

③ $b_{n+1} < b_n$ $\quad \frac{1}{(n+1)^2} < \frac{1}{n^2}$ ✓ $\Rightarrow \boxed{\text{Conv.}}$ by AST

146. $\sum_{n=1}^{\infty}\frac{5^n}{n^2}$ $\boxed{\text{Div.}}$ by n^{th} term Div. Test

$\lim_{n\to\infty}\frac{5^n}{n^2}\neq 0$ because $\lim_{x\to\infty}\frac{5^x}{x^2} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{5^x(\ln 5)}{2x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{5^x(\ln 5)^2}{2} = \infty \neq 0.$

147. $\sum_{n=1}^{\infty}\frac{1}{n+7} \approx \sum\frac{1}{n}$

$\lim_{n\to\infty}\frac{\frac{1}{n+7}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+7} = 1$ non-zero Finite $\Rightarrow$ both series share same behavior

Since $\sum\frac{1}{n}$ diverges, then $\sum\frac{1}{n+7}$ $\boxed{\text{diverges}}$ by L.C.T

$f(x)=\frac{1}{x+7}$ decreasing $f'(x)=\frac{-1}{(x+7)^2}<0$

or can use integral test $\int_1^{\infty}\frac{1}{x+7}dx = \lim_{t\to\infty}\int_1^t\frac{1}{x+7}dx = \lim_{t\to\infty}\ln|x+7|\Big|_1^t = \lim_{t\to\infty}\ln|t+7| - \ln|8| = \infty \Rightarrow$ series $\boxed{\text{diverges}}$ by integral test

148. $\sum_{n=1}^{\infty}\frac{5^n}{2^n+3^n}$ $\boxed{\text{Diverges}}$ by n^{th} term Divergence Test

$\lim_{n\to\infty}\frac{5^n}{2^n+3^n}\cdot\frac{1/3^n}{1/3^n} = \lim_{n\to\infty}\frac{(5/3)^n}{(2/3)^n+1} = \infty$ (Can also use Ratio Test)

or L.C.T with $\sum(5/3)^n$

$\lim_{n\to\infty}\frac{\frac{5^n}{2^n+3^n}}{5^n/3^n} = \lim_{n\to\infty}\frac{5^n}{5^n}\cdot\frac{3^n}{2^n+3^n}\cdot\frac{1/3^n}{1/3^n} = \lim_{n\to\infty}\frac{1}{(2/3)^n+1} = 1$ non-zero Finite, Since $\sum(5/3)^n$ Div $\Rightarrow \sum\frac{5^n}{2^n+3^n}$ Di[v] by LC[T]

149. $\sum_{\cancel{n=1}}^{\infty}\frac{1}{n\sqrt{\ln n}}$

again need $n=2$ here

$f(x)=\frac{1}{x\sqrt{\ln x}}$ decreasing since $f'(x)=\frac{-1}{(x\sqrt{\ln x})^2}\cdot\left[x\frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x}+\sqrt{\ln x}\right]<0$ o.k.

$\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}dx = \lim_{t\to\infty}\int_2^t\frac{1}{x\sqrt{\ln x}}dx = \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{\sqrt{u}}du = \lim_{t\to\infty}2\sqrt{u}\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty}2\sqrt{\ln t} - 2\sqrt{\ln 2} = \infty$

$u=\ln x$ $\quad x=2 \Rightarrow u=\ln 2$

$du=\frac{1}{x}dx$ $\quad x=t \Rightarrow u=\ln t$

$\Rightarrow$ series $\boxed{\text{Diverges}}$ by Integral T[est]

150. $\sum_{n=1}^{\infty}\frac{5\cdot 2^n+6^n}{n2^n} = \sum\frac{5\cdot 2^n}{n2^n}+\sum\frac{6^n}{n2^n} = \underbrace{5\sum\frac{1}{n}}_{\substack{\text{Div. } p=1\\ p\text{-series}}}+\underbrace{\sum\frac{3^n}{n}}_{\text{Div. by } n^{th} \text{ Term Div. Test}}$

$\lim_{n\to\infty}\frac{3^n}{n}=\infty$

Sum of 2 Divergent Series (with all positive terms) is Divergent

try comparison

Since $\frac{5\cdot 2^n+6^n}{n2^n} \geq \frac{5\cdot 2^n}{n2^n} = \frac{5}{n}$

and $5\sum\frac{1}{n}$ $(p=1)$ diverges, then the original (larger) series $\boxed{\text{diverges}}$ by C.T.

151. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^2+1}$ $\boxed{\text{Diverges}}$ by n^{th} term Divergence Test

$\displaystyle\lim_{n\to\infty} \frac{n^2-1}{3n^2+1} = \frac{1}{3} \neq 0.$ terms not approaching 0

152. $\displaystyle\sum_{n=1}^{\infty} \frac{7}{25+n^2}$ Since $\dfrac{7}{25+n^2} \leq \dfrac{7}{n^2}$ and $\underbrace{7\overbrace{\sum \frac{1}{n^2}}^{\substack{\text{converges.}\\ p=2}}}_{\text{constant multiple of convergent is convergent}}$ converges,

then the original (smaller) series $\boxed{\text{Converges}}$ by comparison, C.T.

$(x) = \dfrac{7}{x^2+25}$ decreasing
ince
$'(x) = \dfrac{-14x}{(x^2+25)^2} < 0$

or Integral Test.

$$7\int_1^{\infty} \frac{1}{x^2+25}\,dx = 7\lim_{t\to\infty}\int_1^{t} \frac{1}{x^2+25}\,dx = \frac{7}{5}\lim_{t\to\infty} \arctan\left(\frac{x}{5}\right)\Big|_1^t = \frac{7}{5}\lim_{t\to\infty}\left[\overbrace{\arctan\left(\frac{t}{5}\right)}^{\pi/2} - \underbrace{\arctan\left(\frac{1}{5}\right)}_{\text{finite}}\right] = \text{Finite}$$

$\Rightarrow$ series $\boxed{\text{Conv.}}$ by Integ Test

153. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}}\right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n n!} = \lim_{n\to\infty} \frac{\overset{2}{\cancel{2^{n+1}}}}{\cancel{2^n}}\,\frac{\overset{n+1}{\cancel{(n+1)!}}}{\cancel{n!}}\,\frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)(n+1)^n}} = \lim_{n\to\infty} \frac{2\cancel{(n+1)}}{\cancel{(n+1)}}\overbrace{\left(\frac{n}{n+1}\right)^n}^{1/e} = \frac{2}{e} < 1$$

$\Rightarrow \boxed{\text{Converges}}$ by Ratio Test

154. $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!+4^{n+1}}{(3n+1)!}$

Since $\dfrac{(3n)!+4^{n+1}}{(3n+1)!} \geq \dfrac{(3n)!}{(3n+1)!}$ and

$\displaystyle\sum \frac{\cancel{(3n)!}}{\underbrace{(3n+1)!}_{(3n+1)\,\cancel{(3n)!}}} = \underbrace{\sum \frac{1}{3n+1}}_{\text{Div. by LCT with } \sum\frac{1}{n}}$ is Divergent $\Rightarrow$ original (larger) series $\boxed{\text{Diverges}}$ by C.T.

or

$$= \sum \frac{(3n)!}{(3n+1)!} + \underbrace{\sum \frac{4^{n+1}}{(3n+1)!}}_{\substack{\text{actually}\\ \text{converges}\\ \text{by Ratio Test}}}$$

$$= \underbrace{\sum \frac{1}{3n+1}}_{\substack{\text{Div. by LCT}\\ \text{with } \sum\frac{1}{n}}}$$

$\Rightarrow$ But Sum Div. $\oplus$ Conv. $= \boxed{\text{Div.}}$

Ratio Test on 2^{nd} Piece

$$\lim_{n\to\infty}\left|\frac{\frac{4^{n+2}}{(3n+4)!}}{\frac{4^{n+1}}{(3n+1)!}}\right| = \lim_{n\to\infty} \frac{\overset{4}{\cancel{4^{n+2}}}}{\cancel{4^{n+1}}}\,\frac{\cancel{(3n+1)!}}{\underbrace{(3n+4)!}_{(3n+4)(3n+3)(3n+2)\cancel{(3n+1)!}}} = 0 < 1$$

$\Rightarrow$ Conv. by Ratio Test.

155. $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}}\right| = \lim_{n\to\infty} \frac{n+1}{e^{n+1}}\cdot\frac{e^n}{n}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)}_{1}\cdot\frac{e^n}{e^{n+1}} = \frac{1}{e} < 1$$

$\Rightarrow \boxed{\text{Conv.}}$ by Ratio Test

or more slick use $n < 2^n$ so $\displaystyle\sum \frac{n}{e^n} \leq \sum \frac{2^n}{e^n} = \underbrace{\sum\left(\frac{2}{e}\right)^n}_{\text{conv. geom. } r = 2/e < 1}$

[illegible] by C.T.

156. $\displaystyle\sum_{n=1}^{\infty} \pi^{-n} e^n = \sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ geometric $r = \frac{e}{\pi} < 1 \Rightarrow \boxed{\text{converges}}$ ✓

or use Ratio Test...

157. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n-1)!}$ Ratio Test

$(2(n+1)-1)! = (2n+2-1)! = (2n+1)!$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{(2(n+1)-1)!}}{\frac{n!}{(2n-1)!}}\right| = \lim_{n\to\infty} \frac{\overset{(n+1)\cancel{n!}}{(n+1)!}}{\cancel{n!}} \cdot \frac{\cancel{(2n-1)!}}{\underbrace{(2n+1)!}_{(2n+1)(2n)\cancel{(2n-1)!}}} = \lim_{n\to\infty} \frac{n+1}{4n^2+2n} = 0 < 1$$ ✓

drop |.|

$\Rightarrow \boxed{\text{converges}}$ by Ratio Test

158. $\displaystyle\sum_{n=1}^{\infty} 3 + \frac{1}{3^n}$ $\boxed{\text{Diverges}}$ ✓ by n^{th} term Divergence Test

$$\lim_{n\to\infty} 3 + \cancel{\frac{1}{3^n}}^{\,0} = 3 \neq 0$$

159. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ Ratio Test

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{\overset{(n+1)\cancel{n!}}{(n+1)!}}{\cancel{n!}} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{\cancel{(n+1)}(n+1)^n}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \nearrow \frac{1}{e} = \frac{1}{e} < 1$$ ✓

$\Rightarrow \boxed{\text{Converges}}$ by Ratio Test

160. $\displaystyle\sum_{n=1}^{\infty} e^{\frac{1}{n}}$ $\boxed{\text{Diverges}}$ ✓ by n^{th} term Divergence Test

$$\lim_{n\to\infty} e^{1/n} = e^0 = 1 \neq 0.$$

161. $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ Ratio Test

$$L=\lim_{n\to\infty}\frac{\frac{[(n+1)!]^2}{[2(n+1)]!}}{\frac{(n!)^2}{(2n)!}}=\lim_{n\to\infty}\frac{[(n+1)!]^2}{[2(n+1)]!}\cdot\frac{(2n)!}{(n!)^2}=\lim_{n\to\infty}\frac{[(n+1)!]^2}{[n!]^2}\cdot\frac{(2n)!}{(2n+2)!}=\lim_{n\to\infty}\frac{(n+1)\cancel{n!}\cdot(n+1)\cancel{n!}}{\cancel{n!}\cdot\cancel{n!}}\cdot\frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$=\lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$=\lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n}=\frac{1}{4}<1$$

$\Rightarrow$ $\boxed{\text{converges}}$ by Ratio Test

162. $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n^3 7^n}$ Since $\frac{3}{n^3 7^n}\le\frac{3}{n^3}$ and $\underbrace{\sum\frac{3}{n^3}}_{p=3>1}$ converges.

or $\frac{3}{n^3 7^n}\le\frac{3}{7^n}$ and $\sum\frac{3}{7^n}$ is convergent geometric series $r=1/7<1$

in either comparison case, the original (smaller) series $\boxed{\text{converges}}$ by C.T.

$\boxed{\text{or}}$ use Ratio Test

$$\lim_{n\to\infty}\frac{\frac{3}{(n+1)^3 7^{n+1}}}{\frac{3}{n^3 7^n}}=\lim_{n\to\infty}\frac{\cancel{3}}{(n+1)^3 7^{n+1}}\cdot\frac{n^3 7^n}{\cancel{3}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3\cdot\frac{7^n}{7^{n+1}}=\frac{1}{7}<1\Rightarrow\text{converges by Ratio Test}$$

163. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n n}{(n+1)^2}$ Ratio Test

$$\lim_{n\to\infty}\frac{\frac{2^{n+1}(n+1)}{(n+2)^2}}{\frac{2^n n}{(n+1)^2}}=\lim_{n\to\infty}\frac{2^{n+1}(n+1)}{(n+2)^2}\cdot\frac{(n+1)^2}{2^n n}=\lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\left(\frac{n+1}{n}\right)\cdot\left(\frac{n+1}{n+2}\right)^2=2>1\Rightarrow\boxed{\text{Diverges}}\text{ by Ratio Test.}$$

164. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n n^2}{(n+1)!}$ Ratio Test

$$\lim_{n\to\infty}\frac{\frac{2^{n+1}(n+1)^2}{(n+2)!}}{\frac{2^n n^2}{(n+1)!}}=\lim_{n\to\infty}\frac{2^{n+1}(n+1)^2}{(n+2)!}\cdot\frac{(n+1)!}{2^n\cdot n^2}=\lim_{n\to\infty}\frac{2^{n+1}}{2^n}\left(\frac{n+1}{n}\right)^2\cdot\frac{\cancel{(n+1)!}}{\underbrace{(n+2)!}_{(n+2)\cancel{(n+1)!}}}=\lim_{n\to\infty}\frac{2}{n+2}=0<1$$

$\Rightarrow$ $\boxed{\text{converges}}$ by Ratio Test

165. $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{n!}$ Ratio Test

$$\lim_{n\to\infty}\frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}}=\lim_{n\to\infty}\frac{5^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n}=\lim_{n\to\infty}\frac{5^{n+1}}{5^n}\cdot\frac{n!}{\underbrace{(n+1)!}_{(n+1)\,n!}}=\lim_{n\to\infty}\frac{5}{n+1}=0<1$$

$\Rightarrow$ $\boxed{\text{Converges}}$ by Ratio Test

166. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$ Ratio Test

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{5^{n+1}}}{\frac{n!}{5^n}} = \lim_{n\to\infty} \frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}}{n!} \cdot \frac{5^n}{5^{n+1}}_{\,5} = \lim_{n\to\infty} \frac{n+1}{5} = \infty > 1$$

$\Rightarrow \boxed{\text{Diverges}}$ by Ratio Test ✓

or by n^{th} Term Divergence Test

167. $\displaystyle\sum_{n=2}^{\infty} \left(-\frac{3}{4}\right)^n$ Geometric Series $|r| = |-3/4| = 3/4 < 1$ $\Rightarrow \boxed{\text{Convergent}}$ ✓

168. $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n) = \cos\pi + \cos(2\pi) + \cos(3\pi) + \cdots = -1 + 1 - 1 + \cdots = \sum_{n=1}^{\infty} (-1)^n$ ✓

$\boxed{\text{Diverges}}$ by n^{th} term Divergence Test

since $\lim_{n\to\infty} (-1)^n$ D.N.E.

169. $\displaystyle\sum_{n=2}^{\infty} e^{\left(\frac{\sin n}{n}\right)}$

Recall: $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$, with $-\frac{1}{n} \to 0$, $\frac{\sin n}{n} \to 0$, $\frac{1}{n} \to 0$ $\Rightarrow \lim_{n\to\infty} \frac{\sin n}{n} = 0$ by Squeeze law ✓

$$\lim_{n\to\infty} e^{\frac{\sin n}{n}} = e^{\lim_{n\to\infty} \frac{\sin n}{n} \to 0} = e^0 = 1$$

$\Rightarrow$ Series $\boxed{\text{Diverges}}$ by n^{th} term Divergence Test

170. $\displaystyle\sum_{n=2}^{\infty} \frac{9^n}{(-2)^{n+1}n}$ Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{9^{n+1}}{(-2)^{n+2}(n+1)}}{\frac{9^n}{(-2)^{n+1}n}} \right| = \lim_{n\to\infty} \frac{\cancel{9^{n+1}}^{\,9}}{\cancel{9^n}} \cdot \frac{\cancel{n}^{\,1}}{\cancel{n+1}} \cdot \frac{2^{n+1}}{2^{n+2}}_{\,2} = \frac{9}{2} > 1$$

$\Rightarrow \boxed{\text{Diverges}}$ by Ratio Test ✓

171. $\displaystyle\sum_{n=2}^{\infty} \frac{3 \cdot 7^n - n^6}{n^7 7^n} = \sum_{n=2}^{\infty} \frac{3 \cdot \cancel{7^n}}{n^7 \cdot \cancel{7^n}} - \sum_{n=2}^{\infty} \frac{n^6}{n^7 7^n} = \underbrace{\sum \frac{3}{n^7}}_{\text{convergent p-series } p=7>1} - \underbrace{\sum \frac{1}{n \cdot 7^n}}_{\text{convergent by Ratio Test or Comparison}} \Rightarrow$ Original Series $\boxed{\text{Converges}}$ because it's a Difference of 2 convergent series

$\displaystyle\sum \frac{1}{n \cdot 7^n} \rightarrow \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)7^{n+1}}}{\frac{1}{n \cdot 7^n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{7^n}{7^{n+1}} = \frac{1}{7} < 1$ $\Rightarrow$ converges by Ratio Test

or since $\frac{1}{n7^n} \le \frac{1}{7^n}$ and $\sum \frac{1}{7^n}$ conv. Geometric $(r = 1/7 < 1) \Rightarrow$ smaller series conv. by C.T.

172. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)^n n!}{(2n)!}$ Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(2(n+1))^{n+1}(n+1)!}{[2(n+1)]!}}{\frac{(2n)^n n!}{(2n)!}} \right| = \lim_{n\to\infty} \frac{(2n+2)^{n+1}(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{(2n)^n n!} = \lim_{n\to\infty} \frac{\overbrace{(2n+2)^{n+1}}^{(2n+2)(2n+2)^n}}{(2n)^n} \cdot \frac{\overbrace{(n+1)!}^{(n+1)n!}}{n!} \cdot \frac{(2n)!}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)(2n)!}}$$

$$= \lim_{n\to\infty} \left(\frac{2n+2}{2n}\right)^n \cdot \left(\frac{n+1}{2n+1}\right) = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{2} = \frac{e}{2} > 1 \Rightarrow \boxed{\text{Diverges}} \text{ by Ratio Test}$$

173. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n (n!)^3}{(2n)! n^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{4^{n+1}[(n+1)!]^3}{[2(n+1)]!\,(n+1)^{n+1}}}{\frac{4^n (n!)^3}{(2n)!\, n^n}} \right| = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \cdot \frac{\overbrace{[(n+1)!]^3}^{[(n+1)n!]^3 = (n+1)^3(n!)^3}}{(n!)^3} \cdot \frac{(2n)!}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)(2n)!}} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)(n+1)^n}} = \lim_{n\to\infty} \frac{4 \cdot (n+1)^3}{(2n+2)(2n+1)} \cdot \frac{1}{n+1} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \frac{4(n^2+2n+1)}{4n^2+6n+2} \cdot \frac{1}{e} = \frac{1}{e} < 1$$

$\Rightarrow \boxed{\text{Converges}}$ by Ratio Test.

Even More Series: Determine whether each of the following series **converges absolutely**, **converges conditionally**, or **diverges**. Justify your answers.

174. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n} \rightarrow \sum \frac{1}{5n} = \frac{1}{5} \sum \frac{1}{n}$ Diverges since it's Constant multiple of Divergent $p=1$ series.

Can use AST here

① $b_n = \frac{1}{5n} > 0$

② $\lim_{n\to\infty} \frac{1}{5n} = 0$

③ $b_{n+1} < b_n$ $\quad \frac{1}{5(n+1)} < \frac{1}{5n}$ $\Rightarrow$ converges by AST

Finally, original series is $\boxed{\text{Conditionally Convergent}}$

175. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$ Try Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2} \frac{n+1}{2^{n+1}}}{(-1)^{n+1} \frac{n}{2^n}} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2} < 1 \Rightarrow \boxed{\text{Absolutely Convergent}}$$

by Ratio Test.

176. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{7n-3}$ $\hookrightarrow \sum \frac{1}{7n-3} \approx \underbrace{\sum \frac{1}{n}}_{\text{Div. } p=1}$ $\lim_{n\to\infty} \frac{\frac{1}{7n-3}}{1/n} = \lim_{n\to\infty}\frac{n}{7n-3} = \frac{1}{7}$ non-zero Finite

$\Rightarrow$ Both Share Same Behavior by L.C.T

$\Rightarrow \sum \frac{1}{7n-3}$ Diverges

Try AST

1. $b_n = \frac{1}{7n-3} > 0$

2. $\lim_{n\to\infty}\frac{1}{7n-3} = 0$

3. $b_{n+1} < b_n$ $\frac{1}{7(n+1)-3} < \frac{1}{7n-3} \Leftrightarrow 7n-3 < 7n+4$ ✓ o.k. $\Rightarrow \sum \frac{(-1)^{n+1}}{7n-3}$ converges by AST

$\Rightarrow \boxed{\text{Conditionally Conv.}}$ ✓

177. $\sum_{\cancel{n=1}}^{\infty}(-1)^n\frac{1}{n\ln n}$ put n=2 here $\hookrightarrow \sum \frac{1}{n\ln n}$ try Integral Test

$\int_2^\infty \frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u}du = \lim_{t\to\infty} \ln|u|\Big|_{\ln 2}^{\ln t}$ $u = \ln x$ $du = \frac{1}{x}dx$

$= \lim_{t\to\infty}\left(\ln|\ln t| - \ln|\ln 2|\right)$ $\nearrow \infty$

$= \infty \Rightarrow$ Diverges by Integral Test

AST

1. $b_n = \frac{1}{n\ln n} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n\ln n} = 0$

3. $b_{n+1} < b_n$ $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln n}$ ✓ or $f(x) = \frac{1}{x\ln x}$

$\Rightarrow f'(x) = \frac{-1}{(x\ln x)^2}\left[x\cdot\frac{1}{x} + \ln x\right] < 0$

$\Rightarrow$ Decreasing terms ✓

$\Rightarrow$ converges by AST

$\Rightarrow \boxed{\text{Conditionally Convergent}}$ ✓

178. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\ln(n+1)}$

use: $\ln(n+1) \le n+1$ Since $\frac{1}{\ln(n+1)} \ge \frac{1}{n+1}$ and $\sum\frac{1}{n+1}$ Diverges (why?) then $\sum \frac{1}{\ln(n+1)}$ Diverges by CT

(note. Abs. Conv not helpful)

$\Rightarrow \frac{1}{\ln(n+1)} \ge \frac{1}{n+1}$ $\sum\frac{1}{n+1} \approx \sum\frac{1}{n}$ $\rightsquigarrow \lim_{n\to\infty}\frac{\frac{1}{n+1}}{1/n} = 1$ non-zero finite by LCT (both Diverge)

Then try

1. $b_n = \frac{1}{\ln(n+1)} > 0$

2. $\lim_{n\to\infty}\frac{1}{\ln(n+1)} = 0$

3. $b_{n+1} < b_n$ $\frac{1}{\ln(n+1)} < \frac{1}{\ln n} \Leftrightarrow \ln n < \ln(n+1)$ ($\ln$ increasing)

$\Rightarrow \sum(-1)^{n+1}\frac{1}{\ln n}$ conv. by AST

$\Rightarrow$ Conditionally Convergent ✓

179. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n+3^n}$ $\hookrightarrow \sum \frac{2^n}{n+3^n}$ note $\frac{2^n}{n+3^n} \le \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

Conv. Geom. $r = 2/3 < 1$ and $\sum\left(\frac{2}{3}\right)^n$

$\Rightarrow$ (smaller) $\sum\frac{2^n}{n+3^n}$ converges by C.T.

$\Rightarrow \boxed{\text{Absolute Convergence}}$ ✓ by Absolute Convergence Test

or try Ratio Test

$\lim_{n\to\infty}\left|\frac{\frac{(-1)^n 2^{n+1}}{(n+1)+3^{n+1}}}{\frac{(-1)^{n-1}2^n}{n+3^n}}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{n+3^n}{\underbrace{n+1+3^{n+1}}_{n+1+3\cdot 3^n}}\cdot\frac{\left(\frac{1}{3^n}\right)}{\left(\frac{1}{3^n}\right)}$

$= \lim_{n\to\infty} 2\,\frac{\frac{n}{3^n}+1}{\frac{n+1}{3^n}+3} = \frac{2}{3} < 1 \Rightarrow$ Absolutely Convergence by Ratio Test

180. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{10n+1}$

$\boxed{\text{Diverges}}$ ✓ by n^{th} term Divergence Test

since $\lim_{n\to\infty}\frac{n}{10n+1} = \frac{1}{10}$ then $\lim_{n\to\infty}(-1)^{n+1}\frac{n}{10n+1}$ DNE same as #111. as the Alternating Series

181. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$

$\sum \frac{n}{n^2+1} \approx \sum \frac{1}{n}$ Div. p-series $p=1$ $\Rightarrow \sum \frac{n}{n^2+1}$ also Div. by L.C.T.

still need to examine Absolute Series

$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$ Finite non-zero

try AST

① $b_n = \frac{n}{n^2+1} > 0$

② $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

③ $b_{n+1} < b_n$ $\quad f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} < 0 \Rightarrow$ Decreasing Terms

$\Rightarrow$ conv. by AST.

$\Rightarrow \boxed{\text{Conditionally Convergent}}$ ✓

182. $\displaystyle\sum_{n=1}^{\infty}\frac{\overbrace{\cos(\pi n)}^{(-1)^n}}{n} = \sum \frac{(-1)^n}{n}$ is the Alternating Harmonic Series

which we know is $\boxed{\text{conditionally conv.}}$ ✓

- $\sum \frac{1}{n}$ Divergent Harmonic ($p=1$)
- $\sum (-1)^n \frac{1}{n}$ Converges by AST

183. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln n}{n}$ $\quad \rightarrow \sum \frac{\ln n}{n}$

can use Integral Test (showed earlier, decreasing) $f(x) = \frac{\ln x}{x}$

$\int_1^\infty \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx = \lim_{t\to\infty} \left.\frac{(\ln x)^2}{2}\right|_1^t = \lim_{t\to\infty} \frac{(\ln t)^2}{2} - \frac{(\ln 1)^2}{2} = \infty$

$\Rightarrow$ Diverges by Integral Test

or Note $\sum \frac{\ln n}{n} \geq \sum \frac{1}{n}$ (since $\ln n \geq 1$ for $n \geq 3$)

$\Rightarrow$ Diverges by C.T.

AST

① $b_n = \frac{\ln n}{n} > 0$

② $\lim_{n\to\infty} \frac{\ln n}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

③ $b_{n+1} < b_n$ $\quad f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{x \cdot \frac{1}{x} - \ln x (1)}{x^2} = \frac{1 - \ln x}{x^2} < 0$ for $x > e$ o.k.

$\Rightarrow$ Conv. by AST.

$\Rightarrow \boxed{\text{Conditionally Convergent}}$ ✓

184. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\cosh n} = \sum (-1)^n \frac{1}{\left(\frac{e^n + e^{-n}}{2}\right)} = \sum (-1)^n \cdot \frac{2}{e^n + e^{-n}}$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}\cancel{2}}{e^{n+1}+e^{-n-1}}}{\frac{(-1)^n \cdot \cancel{2}}{e^n + e^{-n}}}\right| = \lim_{n\to\infty} \frac{e^n + e^{-n}}{e^{n+1} + \underbrace{e^{-n-1}}_{\frac{1}{e^{n+1}}}} \cdot \frac{\left(\frac{1}{e^n}\right)}{\left(\frac{1}{e^n}\right)} = \lim_{n\to\infty} \frac{1 + \frac{1}{e^{2n}}}{e + \frac{1}{e^{2n+1}}} = \frac{1}{e} < 1 \Rightarrow \boxed{\text{Absolutely convergent}}$$

by Ratio test ✓

185. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^{n^2}}$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(n+1)!}{2^{(n+1)^2}}}{\frac{(-1)^{n+1}n!}{2^{n^2}}}\right| = \lim_{n\to\infty} \frac{\overbrace{(n+1)!}^{(n+1)n!}}{n!} \cdot \frac{\cancel{2^{n^2}}}{\underbrace{2^{(n+1)^2}}_{\underbrace{2^{n^2+2n+1}}_{\cancel{2^{n^2}} \cdot 2^{2n+1}}}} = \lim_{n\to\infty} \frac{n+1}{2^{2n+1}} \overset{L'H}{=} 0 < 1$$

$\Rightarrow \boxed{\text{Absolute Convergence}}$ ✓

by Ratio test

186. $\displaystyle\sum_{n=2}^{\infty} \frac{n(-3)^{2n+1}}{10^n}$ Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)(-3)^{2(n+1)+1}}{10^{n+1}}}{\frac{n(-3)^{2n+1}}{10^n}} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{3^{2n+3}}{3^{2n+1}} \cdot \frac{10^n}{10^{n+1}} = \frac{9}{10} < 1 \Rightarrow \boxed{\text{Absolute Convergence}}$$

by Ratio Test ✓

187. $\displaystyle\sum_{n=2}^{\infty} 2^{\ln n}\left(\frac{1}{2}\right)^n = \sum_{n=2}^{\infty} \frac{2^{\ln n}}{2^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{2^{\ln(n+1)}}{2^{n+1}}}{\frac{2^{\ln n}}{2^n}} \right| = \lim_{n\to\infty} \frac{2^{\ln(n+1)}}{2^{\ln n}} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{2^{\ln(n+1)-\ln n}}{2} = \lim_{n\to\infty} \frac{2^{\ln\left(\frac{n+1}{n}\right)}}{2} = \frac{1}{2} < 1 \Rightarrow \boxed{\text{Absolute Convergence}}$$

✓ by Ratio Test

188. $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{n^n}$ Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{7^{n+1}}{(n+1)^{n+1}}}{\frac{7^n}{n^n}} \right| = \lim_{n\to\infty} \frac{7^{n+1}}{7^n} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)(n+1)^n}} = \lim_{n\to\infty} \frac{7}{(n+1)}\left(\frac{n}{n+1}\right)^n = 0 < 1 \Rightarrow \boxed{\text{Absolute Convergence}}$$

✓ by Ratio Test

similar to 179 → 189. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n+3^n} = \sum \frac{(-1)^n 2^n}{n+3^n}$ Ratio Test

OR → $\displaystyle\sum \frac{2^n}{n+3^n} \le \sum \frac{2^n}{3^n} = \sum\left(\frac{2}{3}\right)^n$ conv. geom. $r = 2/3 < 1$ ⇒ Absolute Series $\displaystyle\sum \frac{2^n}{n+3^n}$ conv. by C.T.

⇒ $\boxed{\text{Absolute Converg}}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1}}{(n+1)+3^{n+1}}}{\frac{(-2)^n}{n+3^n}} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n+3^n}{(n+1)+3^{n+1}} \cdot \frac{1/3^n}{1/3^n} = \lim_{n\to\infty} 2 \cdot \frac{\frac{n}{3^n}+1}{\frac{n+1}{3^n}+3} = \frac{2}{3} < 1 \Rightarrow \boxed{\text{Absolute Convergence}}$$

by Ratio Test ✓

190. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$ Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{e^{2(n+1)}}{(n+1)^{n+1}}}{\frac{e^{2n}}{n^n}} \right| = \lim_{n\to\infty} \frac{e^{2n+2}}{e^{2n}} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)(n+1)^n}} = \lim_{n\to\infty} \frac{e^2}{n+1} \cdot \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$$

$\Rightarrow \boxed{\text{(Absolute) Convergence}}$ ✓

by Ratio Test

191. $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^{2n+1}}{n10^n}$ Ratio Test

$$\lim_{n\to\infty}\left|\frac{\frac{(-4)^{2(n+1)+1}}{(n+1)\,10^{n+1}}}{\frac{(-4)^{2n+1}}{n\,10^n}}\right| = \lim_{n\to\infty} \frac{\overbrace{4^{2n+3}}^{4^{2n+1}\cdot 4^2}}{4^{2n+1}}\cdot\frac{\cancel{n}^{\,1}}{\cancel{n}+1}\cdot\frac{\cancel{10^n}}{\underbrace{10^{n+1}}_{10}} = \frac{16}{10} > 1 \Rightarrow \boxed{\text{Diverges}} \text{ by Ratio Test}$$

192. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln(n^2)}{n^3} = \sum_{n=1}^{\infty}(-1)^n\,\frac{2\ln n}{n^3} \rightarrow 2\sum\frac{\ln n}{n^3}$

Note $\dfrac{\ln n}{n^3} \le \overset{(\ln n \le n)}{\dfrac{n}{n^3}} = \dfrac{1}{n^2}$ and $\underbrace{2\overbrace{\sum \frac{1}{n^2}}^{p\text{-series } p=2>1}}_{\text{constant multiple of conv.} = \text{conv}}$ conv.

$\Rightarrow$ (smaller) $2\sum\frac{\ln n}{n^3}$ converges by CT

could also use Integral Test

$\Rightarrow \boxed{\text{Absolute Convergence}}$ ✓

since Absolute Series Convergent.

(by Absolute Convergence Test)

$\Rightarrow$ Original series converges

193. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\arctan n}{n+2^n} \longrightarrow \sum\frac{|\arctan n|}{n+2^n}$

note: $\dfrac{|\arctan n|}{n+2^n} \le \dfrac{\pi/2}{n+2^n} \le \dfrac{\pi}{2}\cdot\left(\dfrac{1}{2^n}\right)$

and $\underbrace{\frac{\pi}{2}\underbrace{\sum\frac{1}{2^n}}_{\text{conv. Geometric } |r|=1/2<1}}_{\text{constant multiple of conv.} = \text{conv.}}$

$\Rightarrow \sum\frac{|\arctan n|}{n+2^n}$ conv. by C.T.

$\Rightarrow \boxed{\text{Absolute Convergence}}$ ✓

194. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n(n+1)} \longrightarrow \sum\frac{1}{n(n+1)}$ = telescoping series $\sum\frac{1}{n}-\frac{1}{n+1} = 1$

$\Rightarrow \boxed{\text{absolute convergence.}}$ ✓

by Absolute Convergence Test

195. $\displaystyle\sum_{n=1}^{\infty}\frac{(n+2)!}{3^n(n!)^2}$ Ratio Test

$$\lim_{n\to\infty}\left|\frac{\frac{[(n+1)+2]!}{3^{n+1}[(n+1)!]^2}}{\frac{(n+2)!}{3^n(n!)^2}}\right| = \lim_{n\to\infty}\frac{\overset{(n+3)\cancel{(n+2)!}}{(n+3)!}}{\cancel{(n+2)!}}\cdot\frac{\cancel{3^n}}{\underset{3}{3^{n+1}}}\cdot\frac{\cancel{(n!)^2}}{\underset{(n+1)^2\cancel{(n!)^2}}{[(n+1)!]^2}} = \lim_{n\to\infty}\frac{n+3}{3(n+1)^2} = 0 < 1$$

✓

$\Rightarrow \boxed{\text{Absolute Convergence}}$

by Ratio Test

Sequences: For each of the following sequences, decide whether it converges or diverges. If it converges, compute its limit.

79. $\left\{\dfrac{1+n-7n^4}{3n^4+8n^3+9}\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty}\dfrac{1+n-7n^4}{3n^4+n^3+9}\dfrac{(1/n^4)}{(1/n^4)}=\lim_{n\to\infty}\dfrac{\frac{1}{n^4}+\frac{1}{n^3}-7}{3+\frac{1}{n}+\frac{9}{n^4}}=\boxed{-7/3}$ converges.

80. $\left\{\dfrac{2^n}{n!}\right\}_{n=1}^{\infty}$

$\dfrac{2^n}{n!}=\dfrac{2\cdot2\cdot2\cdots2\cdot2\cdot2}{1\cdot2\cdot3\cdots(n-2)(n-1)n}\le\dfrac{2\cdot2}{n}=\dfrac{4}{n}$ Squeeze. $0\le\lim_{n\to\infty}\dfrac{2^n}{n!}\le\lim_{n\to\infty}\dfrac{4}{n}=\boxed{0}$ Converges.

81. $\left\{\dfrac{n!}{3^n}\right\}_{n=1}^{\infty}$

$\dfrac{n!}{3^n}=\dfrac{n(n-1)\cdots4\,3\,2\,1}{3\,3\cdots3\,3\,3\,3}\ge\dfrac{2n}{27}$ $\lim_{n\to\infty}\dfrac{n!}{3^n}\ge\lim_{n\to\infty}\dfrac{2}{27}n=\boxed{\infty}$ Diverges

82. $\left\{\dfrac{\sqrt{n}}{\ln n}\right\}_{n=1}^{\infty}$

$\lim_{x\to\infty}\dfrac{\sqrt{x}}{\ln x}\overset{\infty/\infty}{=}\lim_{x\to\infty}\dfrac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}}=\lim_{x\to\infty}\dfrac{x}{2\sqrt{x}}=\lim_{x\to\infty}\dfrac{\sqrt{x}}{2}=\boxed{\infty}$ Diverges

83. $\left\{\ln\left(\dfrac{3n}{n+1}\right)\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty}\ln\left(\dfrac{3n}{n+1}\right)=\ln\left(\lim_{n\to\infty}\dfrac{3n}{n+1}\right)=\boxed{\ln 3}$ converges.

84. $\left\{\dfrac{n^2\sin n}{n^5+7}\right\}_{n=1}^{\infty}$

$0\le\dfrac{n^2\sin n}{n^5+7}\le\dfrac{n^2}{n^5+7}=\dfrac{1}{n^3}$

$0 \quad 0 \quad \boxed{0}$ Converges by Squeeze Law

85. $\left\{\dfrac{1}{3n+7}\right\}_{n=1}^{\infty}$

converges $\lim_{n\to\infty}\dfrac{1}{3n+7}=\boxed{0}$

86. $\{\ln(n^2-7)-\ln(3n^2+n+9)\}_{n=1}^{\infty}$

$\lim_{n\to\infty}\ln(n^2-7)-\ln(3n^2+n+9)=\lim_{n\to\infty}\ln\left(\dfrac{n^2-7}{3n^2+n+9}\right)=\boxed{\ln\frac{1}{3}}$ converge

87. $\{\arctan(n^2+1)\}_{n=1}^{\infty}$

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$\lim_{n\to\infty}\arctan(n^2+1)=\arctan\left(\lim_{n\to\infty}n^2+1\right)=\boxed{\pi/2}$ converges

88. $\{e^{-2n}\}_{n=1}^{\infty}$ $\lim_{n\to\infty}e^{-2n}=\boxed{0}$ conv.

89. $\left\{\dfrac{4}{\ln n}\right\}_{n=1}^{\infty}$ $\lim_{n\to\infty}\dfrac{4}{\ln n}=\boxed{0}$ conv.

90. $\left\{\dfrac{\ln n}{n}\right\}_{n=1}^{\infty}$ $\lim_{x\to\infty}\dfrac{\ln x}{x}\overset{\infty/\infty}{=}\lim_{x\to\infty}\dfrac{1/x}{1}=\boxed{0}$ conv.

91. $\left\{\dfrac{\sqrt{n}}{(\ln n)^2}\right\}_{n=1}^{\infty}$

$\lim_{x\to\infty}\dfrac{\sqrt{x}}{(\ln x)^2}\overset{L'H}{=}\lim_{x\to\infty}\dfrac{\frac{1}{2\sqrt{x}}}{2\ln x\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\dfrac{x}{4\sqrt{x}\ln x}=\lim_{x\to\infty}\dfrac{\sqrt{x}}{4\ln x}=\lim_{x\to\infty}\dfrac{\frac{1}{2\sqrt{x}}}{4/x}=\lim_{x\to\infty}\dfrac{\sqrt{x}}{8}=\boxed{\infty}$ Diverges

92. $\left\{(e^n+n)^{\frac{1}{n}}\right\}_{n=1}^{\infty}$

$\lim_{x\to\infty}(e^x+x)^{1/x}=e^{\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}}\overset{L'H}{=}e^{\lim_{x\to\infty}\frac{\frac{1}{e^x+x}(e^x+1)}{1}}=e^{\lim_{x\to\infty}\frac{e^x+1}{e^x+x}}\overset{L'H}{=}e^{\lim_{x\to\infty}\frac{e^x}{e^x+1}}\ \tfrac{\infty}{\infty}\overset{L'H}{=}e^{\lim_{x\to\infty}\frac{e^x}{e^x}}=\boxed{e^1}$ conv.

93. $\left\{n^{\frac{1}{n}}\right\}_{n=1}^{\infty}$

$\lim_{x\to\infty}x^{1/x}=e^{\lim_{x\to\infty}\frac{1}{x}\ln x}\overset{L'H}{=}e^{\lim_{x\to\infty}\frac{1/x}{1}}=e^0=\boxed{1}$ conv.

94. $\left\{\dfrac{\sin^2 n}{n^2+3}\right\}_{n=1}^{\infty}$

$0\le\dfrac{\sin^2 n}{n^2+3}\le\dfrac{1}{n^2+3}$

$0 \quad 0 \quad \boxed{0}$ by Squeeze Law.

95. $\left\{n\cos\left(\dfrac{1}{n}\right)\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty}n\cos(1/n)=\boxed{\infty}$ Diverges