Math 12, Fall 2006

Solutions to the Final Exam

1(a). $\displaystyle\int_0^4 \frac{dx}{(9+x^2)^{3/2}} = \qquad (x = 3\tan\theta,\ dx = 3\sec^2\theta\, d\theta) \qquad = \int_*^{**} \frac{3\sec^2\theta\, d\theta}{27\sec^3\theta} = \frac{1}{9}\int_*^{**} \cos\theta\, d\theta$

$\displaystyle = \frac{1}{9}\sin\theta\Big|_*^{**} = \frac{1}{9}\cdot\frac{x}{\sqrt{9+x^2}}\Big|_0^4 = \frac{4}{9\sqrt{25}} - 0 = \frac{4}{45},$

where the third-to-last equality is by a right triangle with angle θ, height x, base 3, and therefore hypoteneuse $\sqrt{9+x^2}$.

1(b). $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \qquad (u = \ln x,\ du = x^{-1}\,dx) \qquad = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$

$= \arcsin(\ln x) + C.$

1(c). Applying partial fractions to $\displaystyle\int \frac{12}{x^2(x-2)}\,dx$, we rewrite the integrand as $A/x + B/x^2 + C/(x-2)$. Adding those fractions and comparing the numerator to the original numerator of 12, we get

$$Ax(x-2) + B(x-2) + Cx^2 = 12, \qquad \text{i.e.,} \qquad (A+C)x^2 + (B-2A)x - 2B = 12.$$

Either by evaluating the first equality at $x = 0$ and $x = 2$ and, say, $x = 1$, or by solving the three equations $A+C = 0$, $B-2A = 0$, $-2B = 12$ given by the second equality, we get $A = -3$, $B = -6$, $C = 3$. So the integral is

$$\int \frac{12}{x^2(x-2)}\,dx = \int \frac{-3}{x} - \frac{6}{x^2} + \frac{3}{x-2}\,dx = \frac{6}{x} + 3\ln|x-2| - 3\ln|x| + C.$$

2.(a) Write $\displaystyle\int_{-\infty}^0 \frac{dx}{1+4x^2} = \lim_{t\to-\infty}\int_t^0 \frac{dx}{1+4x^2}$ because the integral is improper at $-\infty$ and nowhere else.

The integral with t is $\displaystyle\int_t^0 \frac{dx}{1+4x^2} = \qquad (u = 2x,\ du = 2\,dx) \qquad = \frac{1}{2}\int_{2t}^0 \frac{du}{1+u^2}$

$\displaystyle = \frac{1}{2}\arctan u\Big|_{2t}^0 = 0 - \frac{1}{2}\arctan(2t).$

So the original integral is $\displaystyle\int_{-\infty}^0 \frac{dx}{1+4x^2} = \lim_{t\to-\infty} -\frac{1}{2}\arctan(2t) = -\frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{4}.$

2(b). Write $\displaystyle\int_{-1}^3 \frac{dx}{x^4} = \int_{-1}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$, since the integral is improper at 0 (in the middle).

The first of these two integrals is $\displaystyle\int_{-1}^0 \frac{dx}{x^4} = \lim_{t\to0^-}\int_{-1}^t \frac{dx}{x^4} = \lim_{t\to0^-} -\frac{1}{3x^3}\Big|_{-1}^t = \lim_{t\to0^-} -\frac{1}{3t^3} - \frac{1}{3} = \infty.$

Thus, this integral, and hence the original integral also, diverges.

3(a). $\displaystyle\lim_{x\to0}\frac{\cosh x - 1}{x^2} = \frac{0}{0}\ \text{L'H} = \lim_{x\to0}\frac{\sinh x}{2x} = \frac{0}{0}\ \text{L'H} = \lim_{x\to0}\frac{\cosh x}{2} = \frac{1}{2}.$

3(b). Write $\lim_{x\to 0^+} (\cos x)^{1/x^2} = \exp\left(\lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2}\right)$ because the original limit is a 1^∞ indeterminate form.

The limit inside the exponential is $\lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2} = \frac{0}{0}$ L'H $= \lim_{x\to 0^+} \frac{-\frac{\sin x}{\cos x}}{2x} = \lim_{x\to 0^+} -\frac{\sin x}{x}\cdot\frac{\cos x}{2}$ $= -1\cdot\frac{1}{2} = -\frac{1}{2}$. So the original limit is $\exp(-1/2) = e^{-1/2}$.

4(a). For the series $\sum_{n=2}^{\infty} \frac{\sin^2(2n)}{2^n}$, note that $0 \le \frac{\sin^2(2n)}{2^n} \le \frac{1}{2^n}$. Since $\sum \frac{1}{2^n}$ converges by the Geometric Series Test, the original series therefore converges by the Comparison Test.

4(b). Applying the Ratio Test to $\sum_{n=1}^{\infty} \frac{(2n)!}{n^n}$, we get

$$\lim_{n\to\infty} \left| \frac{\frac{(2n+2)!}{(n+1)^{n+1}}}{\frac{(2n)!}{n^n}} \right| = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{n+1}\cdot\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{2(2n+1)}{\left(1+\frac{1}{n}\right)^n} = \frac{2\cdot\infty}{e} = \infty > 1,$$

so the series diverges by the Ratio Test.

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5. The series $\sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{n}}{n+68}$ is alternating, and we check $\lim_{n\to\infty} \frac{\sqrt{n}}{n+68} = \lim_{n\to\infty} \frac{n^{-1/2}}{1+68n^{-1}} = 0$.

Moreover, the terms decrease for $n \ge 68$, because if we let $f(x) = \frac{\sqrt{x}}{x+68}$, then

$$f'(x) = \frac{(1/2)x^{-1/2}(x+68) - \sqrt{x}}{(x+68)^2} = \frac{68-x}{2\sqrt{x}(x+68)^2} < 0 \qquad \text{for } x > 68.$$

Thus, the series converges by the Alternating Series Test.

Meanwhile, we can compare the absolute series $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n+68}$ to $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ by LCT:

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+68}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{n}{n+68} = \lim_{n\to\infty} \frac{1}{1+68n^{-1}} = 1 > 0,$$

so the two series are comparable. Since $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by the p-test, the original absolute series diverges by the Limit Comparison Test.

Thus, the original series **converges conditionally**.

6(a). $\sum_{n=1}^{\infty} \frac{(-1)^n\cdot 2^{n+2}}{3^n}$ is a geometric series with initial term $a = -8/3$ and ratio $r = -2/3$. Thus, the sum is $a/(1-r) = (-8/3)/(5/3) = -8/5$.

6(b). $\sum_{n=0}^{\infty} \frac{(-1)^n\pi^{2n+1}}{6^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n(\pi/6)^{2n+1}}{(2n+1)!} = \sin(\pi/6) = \frac{1}{2}$.

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7. Applying the Ratio Test to $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n \cdot 3^n}$ gives

$$\lim_{n\to\infty} \left| \frac{\frac{(x+2)^{n+1}}{(n+1)3^{n+1}}}{\frac{(x+2)^n}{n\cdot 3^n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{|x+2|}{3} = \lim_{n\to\infty} \frac{1}{1+n^{-1}} \cdot \frac{|x+2|}{3} = \frac{|x+2|}{3}$$

The series converges if this limit is < 1 and diverges if the limit is > 1. Expanding $|x+2|/3 < 1$ gives $-3 < x+2 < 3$, i.e., $-5 < x < 1$.

Checking the endpoints, observe that at $x = 1$ the series is $\sum \frac{3^n}{n3^n} = \sum \frac{1}{n}$, which diverges by the p-test. At $x = -5$, the series is $\sum \frac{(-1)^n}{n}$, which is alternating, and the absolute terms $1/n$ decrease with limit 0; so the series converges by the Alternating Series Test.

Thus, the interval of convergence is $[-5, 1)$.

8(a). To find the Maclaurin series for $g(x) = 3xe^{x^4}$, start with $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so that

$e^{x^4} = \sum_{n=0}^{\infty} \frac{x^{4n}}{n!}$, and therefre the desired series is $g(x) = 3xe^{x^4} = \sum_{n=0}^{\infty} \frac{3x^{4n+1}}{n!}$.

8(b). To find the Maclaurin series for $f(x) = \frac{x}{(1+2x)^2}$, start with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and differentiate to get $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$. Substituting $-2x$ for x gives

$\frac{1}{(1+2x)^2} = \sum_{n=1}^{\infty} n \cdot (-2)^{n-1} x^{n-1}$, so that the desired series is

$f(x) = \frac{x}{(1+2x)^2} = \sum_{n=1}^{\infty} n \cdot (-2)^{n-1} x^n$.

9(a). (See me for a sketch of the region R beneath the graph of $y = e^x$ and above the x-axis, for $0 \le x \le 2$).

Slice the solid formed by revolving R about the line $y = -1$ with vertical slices. The slice at position x is an annulus with inner radius $0-(-1) = 1$, and with outer radius $e^x-(-1) = e^x+1$. The slices run from $x = 0$ to $x = 2$. Thus, the volume is

$$\int_0^2 \pi(e^x+1)^2 - \pi(1)^2\, dx = \int_0^2 \pi(e^{2x} + 2e^x)\, dx = \pi \left[\frac{1}{2}e^{2x} + 2e^x\right]_0^2 = \frac{\pi}{2}(e^4 + 4e^2 - 5).$$

9(b). Slice the solid formed by revolving R about the line $x = 4$ into vertical cylinders. The slice passing through position x has radius $4 - x$ and height e^x; the outermost cylinder is at $x = 0$ and the innermost is at $x = 2$. Thus, the volume is

$$\int_0^2 2\pi(4-x)e^x\, dx = \qquad (u = 4-x,\ dv = 2\pi e^x\, dx,\ du = -dx,\ v = 2\pi e^x)$$

$$= 2\pi(4-x)e^x\Big|_0^2 + \int_0^2 2\pi e^x\, dx = 2\pi(2e^2 - 4) + 2\pi \left[e^x\right]_0^2 = 2\pi(3e^2 - 5).$$

10(a). For the parametric curve C given by $x = t + t^2$, $y = t - t^2$ for $0 \le t \le 2$, we compute $x'(t) = 1 + 2t$ and $y'(t) = 1 - 2t$, so that $\frac{dy}{dx} = \frac{1-2t}{1+2t}$.

10(b). See me for a sketch. Roughly speaking, since dy/dx is positive for $t < 1/2$ and negative for $t > 1/2$ (hitting zero at $t = 1/2$ but never hitting ∞), and since (x, y) is $(0, 0)$ for $t = 0$, is $(3/4, 1/4)$ for $t = 1/2$, and is $(6, -2)$ for $t = 2$, we can describe it as follows. The curve starts from the origin, slopes up to the upper right towards a peak at $(3/4, 1/4)$, then slopes down to the lower right to the point $(6, -2)$.

Besides starting at $(0, 0)$, the curve never again hits the y-axis, since $x(t) > 0$ for $t > 0$. However, since $y = 0$ when $t = 0, 1$, and since $x(1) = 2$, the curve crosses the x-axis on the way down at the point $(2, 0)$.

10(c). Since $(x'(t))^2 + (y'(t))^2 = (1 + 2t)^2 + (1 - 2t)^2 = 2 + 8t^2$, the integral for the length of C is $\int_0^2 \sqrt{2 + 8t^2}\, dt$.

10(d). Similarly, the integral for the surface area generated by revolving C about the x-axis is $\int_0^2 2\pi(t - t^2)\sqrt{2 + 8t^2}\, dt$.

11(a). See me for a sketch the curve $r = \sin(3\theta)$. (It is a 3-leaf rose, with one petal pointing down, and the other two pointing into the first and second quadrants, at angles $\pi/6$ and $5\pi/6$. See, for example, the book's solution to 11.4 #13 (in the answer appendix, page A95), but rotated 90 degrees clockwise.)

11(b). To find the area enclosed, it's important to realize that the curve is traced out in full as θ runs only from 0 to π. Thus, the area enclosed is

$$\int_0^\pi \frac{1}{2}\sin^2(3\theta)\, d\theta = \frac{1}{4}\int_0^\pi 1 - \cos(6\theta)\, d\theta = \frac{\theta}{4} - \frac{1}{24}\sin(6\theta)\Big|_0^\pi = \frac{\pi}{4} - 0 - 0 + 0 = \frac{\pi}{4}.$$