Math 12 extra Partial Fraction examples

1. Compute $\displaystyle\int \frac{x^2+7}{x^2+2x+10}\,dx$

Note that the integrand is an improper rational function because the degree of the numberator is not strictly less than that of the denominator. We apply long division of polynomials to write the integrand as a (simple) polynomial and a proper rational piece. That is the degree of the numerator is strictly less than that of the denominator.

Long division yields:

$$\begin{array}{r} 1 \\ x^2+2x+10\overline{)x^2+7} \\ \underline{-(x^2+2x+10)} \\ -2x-3 \end{array}$$

That is $x^2+7=(x^2+2x+10)(1)+(-2x-3)$. Here the remainder term is $-2x-3$. Dividing both sides by $x^2+2x+10$ yields $\displaystyle\frac{x^2+7}{x^2+2x+10}=1+(-2x-3)$

Now $\displaystyle\int \frac{x^2+7}{x^2+2x+10}\,dx=\int 1+\frac{-2x-3}{x^2+2x+10}\,dx=\int 1-\frac{2x+3}{x^2+2x+10}\,dx$

The second piece contains a quadratic irreducible, so partial fractions will not be helpful here. *Complete the square* on that irreducible in order to convert it into an "almost" arctan integral.

$=\displaystyle\int 1-\frac{2x+3}{(x+1)^2+9}\,dx=\int 1\,dx-\int\frac{2(u-1)+3}{u^2+9}\,du$ • see subst. below

$=\displaystyle\int 1\,dx-\int\frac{2u-2+3}{u^2+9}\,du=\int 1\,dx-\int\frac{2u+1}{u^2+9}\,du$

$=\displaystyle\int 1\,dx-\int\frac{2u}{u^2+9}\,du-\int\frac{1}{u^2+9}\,du$ • after split of integrals

• here we have a simple term, a natural log term, and an arctan-ish term

• make sure that you understand how to compute the last two integrals quickly

$=x-\ln|u^2+9|-\frac{1}{3}\arctan\left(\frac{u}{3}\right)+C=\boxed{x-\ln|(x+1)^2+9|-\frac{1}{3}\arctan\left(\frac{x+1}{3}\right)+C}$

We ("invertedly") substituted above $\boxed{\begin{array}{l} u=x+1\Rightarrow x=u-1 \\ du=dx \end{array}}$

2. Compute $\displaystyle\int \frac{x+13}{x(x^2+4x+13)}\,dx$

Note that the integrand is already a proper rational function. The denominator is already factored into a linear factor and a quadratic irreducible factor. (why is that irreducible?) We use the following Partial Fractions decomposition:

$$\frac{x+13}{x(x^2+4x+13)} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+13}$$

Clearing the denominator yields:

$x + 13 = A(x^2 + 4x + 13) + (Bx + C)x$
$x + 13 = Ax^2 + 4Ax + 13A + Bx^2 + Cx$
$x + 13 = (A + B)x^2 + (4A + C)x + 13A$
so that $\boxed{A + B = 0}$ and $\boxed{4A + C = 1}$ and $\boxed{13A = 13}$

Solve for $\boxed{A = 1}$ and $\boxed{B = -1}$ and $\boxed{C = -3}$

Finally, we have decomposed the original integrand. Hopefully this new, but equal, decomposition is *easier* to integrate.

$$\frac{x+13}{x(x^2+4x+13)} = \frac{1}{x} + \frac{-x-3}{x^2+4x+13}$$

Now,

$$= \int \frac{x+13}{x(x^2+4x+13)}\,dx = \int \frac{1}{x} + \frac{-x-3}{x^2+4x+13}\,dx = \int \frac{1}{x} - \frac{x+3}{x^2+4x+13}\,dx$$

• complete the square on the quadratic irreducible second term

$$= \int \frac{1}{x} - \frac{x+3}{(x+2)^2+9}\,dx = \int \frac{1}{x}\,dx - \int \frac{(u-2)+3}{u^2+9}\,du \quad \bullet \text{ see subst. below}$$

$$= \int \frac{1}{x}\,dx - \int \frac{u+1}{u^2+9}\,du = \int \frac{1}{x}\,dx - \int \frac{u}{u^2+9}\,du - \int \frac{1}{u^2+9}\,du \quad \bullet \text{ after split of integrals}$$

• here we have a simple term, a natural log term, and an arctan-ish term

• make sure that you understand how to compute the last two integrals quickly

$$= \ln|x| - \frac{1}{2}\ln|u^2+9| - \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C = \boxed{\ln|x| - \frac{1}{2}\ln|(x+2)^2+9| - \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C}$$

We ("invertedly") substituted above $\boxed{\begin{array}{l} u = x+2 \Rightarrow x = u-2 \\ du = dx \end{array}}$