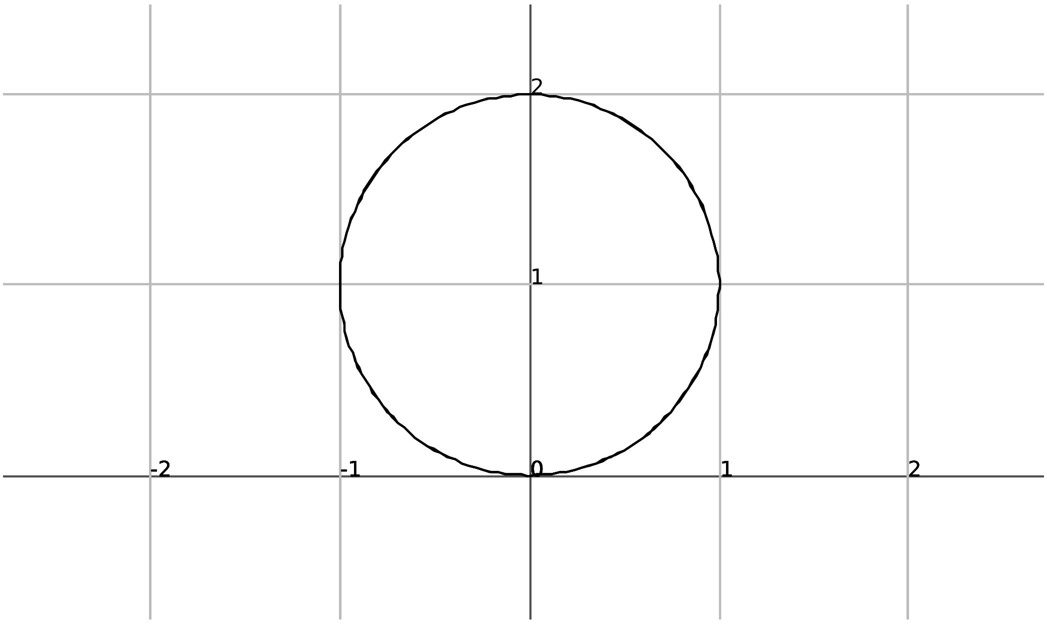
Review Packet for Final Exam

Material since Exam #3

Math 12-D. Benedetto

Polar Coordinates: For each problem, sketch the polar curve(s) and answer the related question(s).

1. Find the area bounded by $r = 2\sin\theta$.



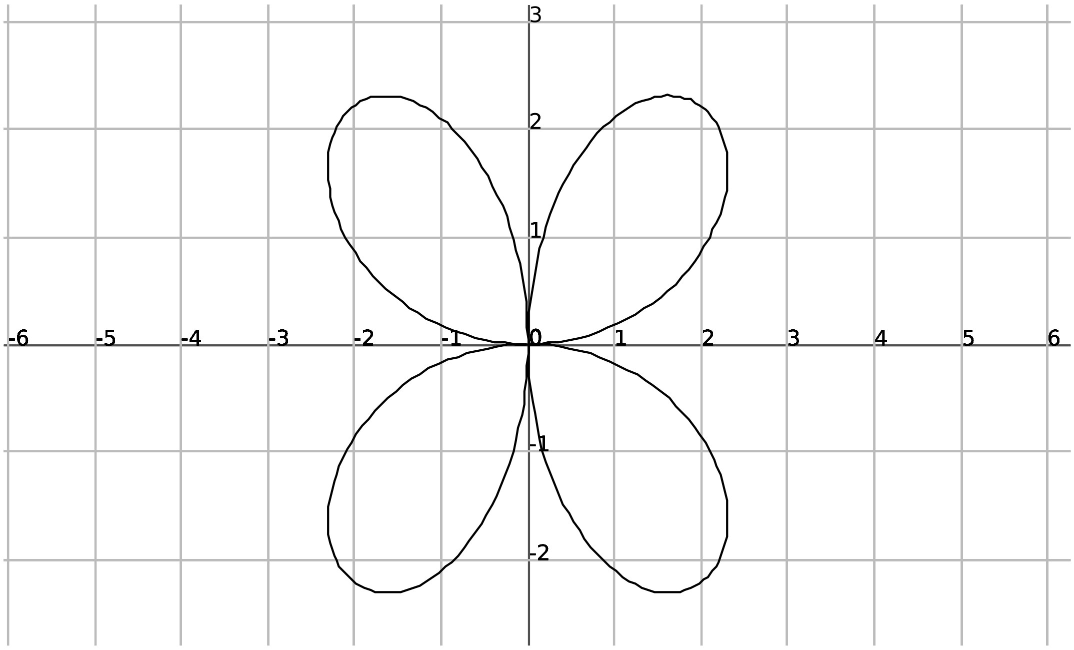
First note that this is just a circle of radius 1, so the area should be π. We will use the area formula for polar curves to double check this. It's important to note that one cycle of the polar circle closes up on itself as θ ranges from $\theta = 0$ to just $\theta = \pi$.

Area $= A = \dfrac{1}{2}\displaystyle\int_0^\pi r^2\ d\theta = \dfrac{1}{2}\int_0^\pi (2\sin\theta)^2\ d\theta = \dfrac{1}{2}\int_0^\pi 4\sin^2\theta\ d\theta = 2\int_0^\pi \sin^2\theta\ d\theta$

$= 2\displaystyle\int_0^\pi \frac{1-\cos(2\theta)}{2}\ d\theta = \int_0^\pi 1-\cos(2\theta)\ d\theta = \theta - \frac{\sin(2\theta)}{2}\Bigg|_0^\pi = \left(\pi - \frac{\sin(2\pi)}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)$

$= \boxed{\pi}$

2. Find the area bounded by one petal loop of the 4 leaved rose $r = 3\sin(2\theta)$.



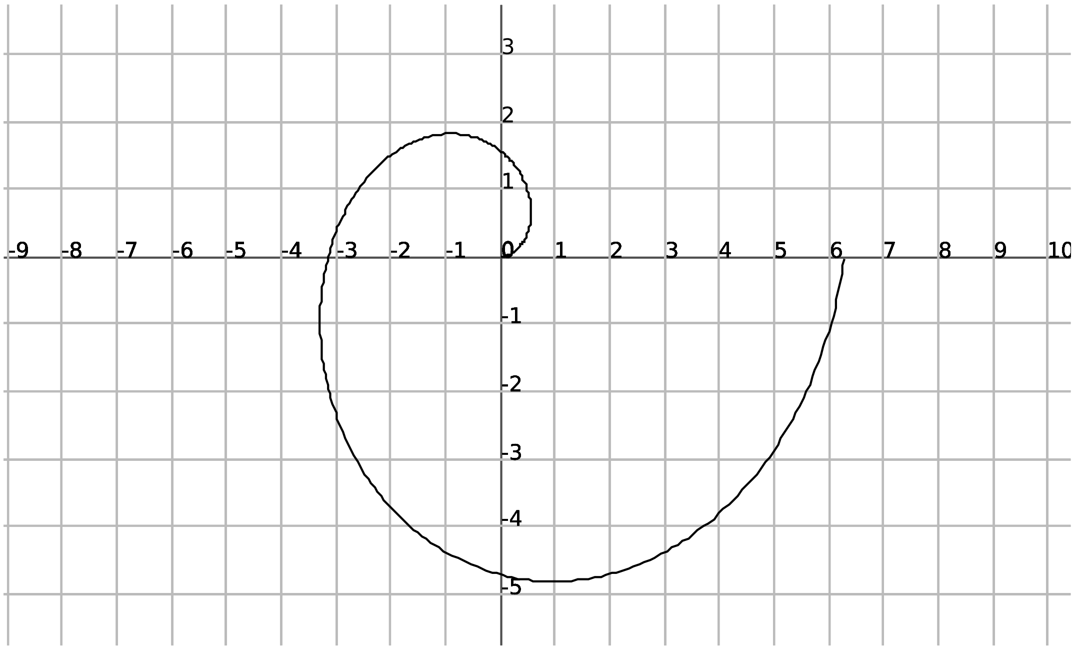
It's important to note that one loop of the polar petal closes up on itself as θ ranges from $\theta = 0$ to just $\theta = \frac{\pi}{2}$.

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (3\sin(2\theta))^2\ d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2(2\theta)\ d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} \sin^2(2\theta)\ d\theta$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2}\ d\theta = \frac{9}{4}\int_0^{\frac{\pi}{2}} 1-\cos(4\theta)\ d\theta = \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Bigg|_0^{\frac{\pi}{2}}$$

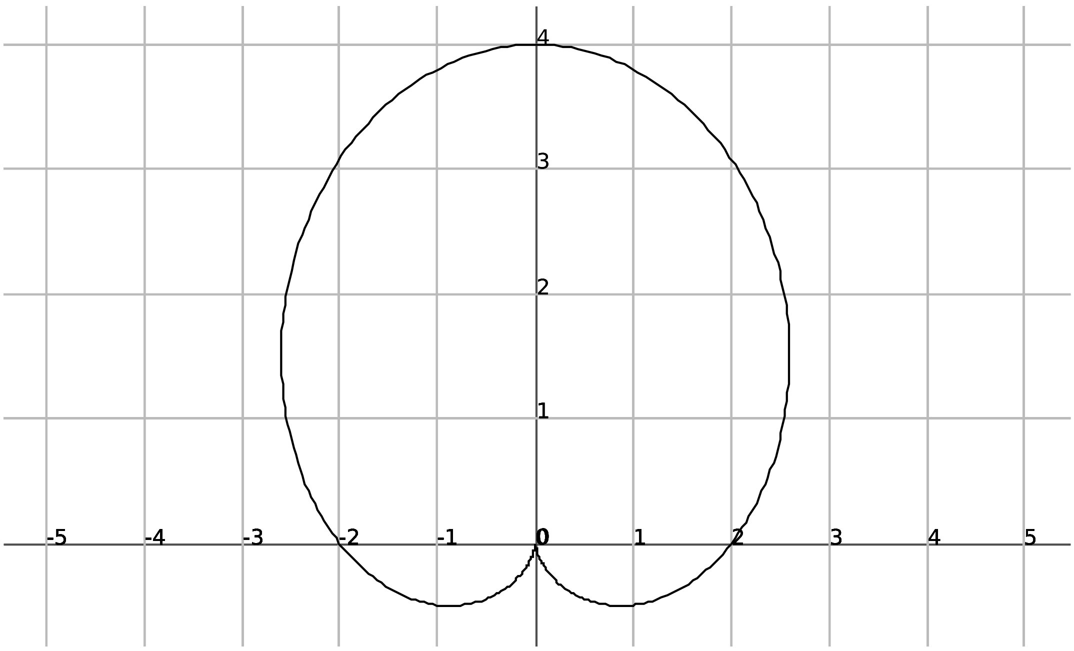
$$= \frac{9}{4}\left(\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) - \left(0 - \frac{\sin 0}{2}\right)\right) = \boxed{\frac{9\pi}{8}}$$

3. Find the area bounded by $r = \theta$ with $0 \le \theta \le 2\pi$



$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \theta^2\, d\theta = \frac{1}{2}\left(\frac{\theta^3}{3}\right)\Big|_0^{2\pi} = \frac{1}{2}\left(\frac{(2\pi)^3}{3} - 0\right) = \frac{1}{2}\left(\frac{8\pi^3}{3}\right) = \boxed{\frac{4\pi^3}{3}}$$

4. Find the area bounded by the cardioid $r = 2 + 2\sin\theta$.



It's important to note that one full cycle of the cardioid closes up on itself as θ ranges from $\theta = 0$ to $\theta = 2\pi$.

$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (2 + 2\sin(\theta))^2\, d\theta = \frac{1}{2}\int_0^{2\pi} 4 + 8\sin\theta + 4\sin^2\theta\, d\theta$$

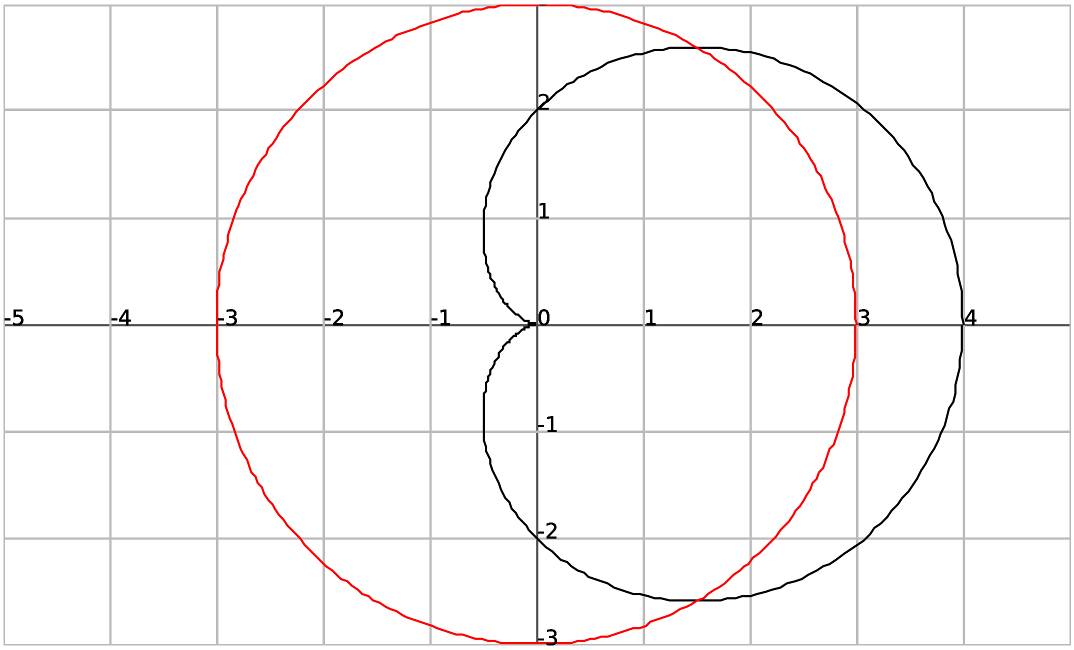
$$= \frac{1}{2}\int_0^{2\pi} 4 + 8\sin\theta + 4\left(\frac{1-\cos(2\theta)}{2}\right)\ d\theta = \frac{1}{2}\int_0^{2\pi} 4 + 8\sin\theta + 2\left(1-\cos(2\theta)\right)\ d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 6 + 8\sin\theta - 2\cos(2\theta)\ d\theta = \frac{1}{2}\left(6\theta - 8\cos\theta - \sin(2\theta)\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2}\left((12\pi - 8\cos(2\pi) - \sin(4\pi)) - (0 - 8\cos 0 - \sin 0)\right)$$

$$\frac{1}{2}\left((12\pi - 8 - 0) - (0 - 8 - 0)\right) = \frac{1}{2}(12\pi - 8 + 8) = \boxed{6\pi}$$

5. Find the area bounded inside $r = 2 + 2\cos\theta$ and outside $r = 3$.



These two polar curves intersect when $2 + 2\cos\theta = 3 \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = -\frac{\pi}{3}$ or $\theta = \frac{\pi}{3}$. Using symmetry, we will integrate from $\theta = 0$ to $\theta = \frac{\pi}{3}$ and double that area.

$$\text{Area} = A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2\right)\ d\theta\right) = \int_0^{\frac{\pi}{3}} \left((2+2\cos\theta)^2 - 3^2\right)\ d\theta$$

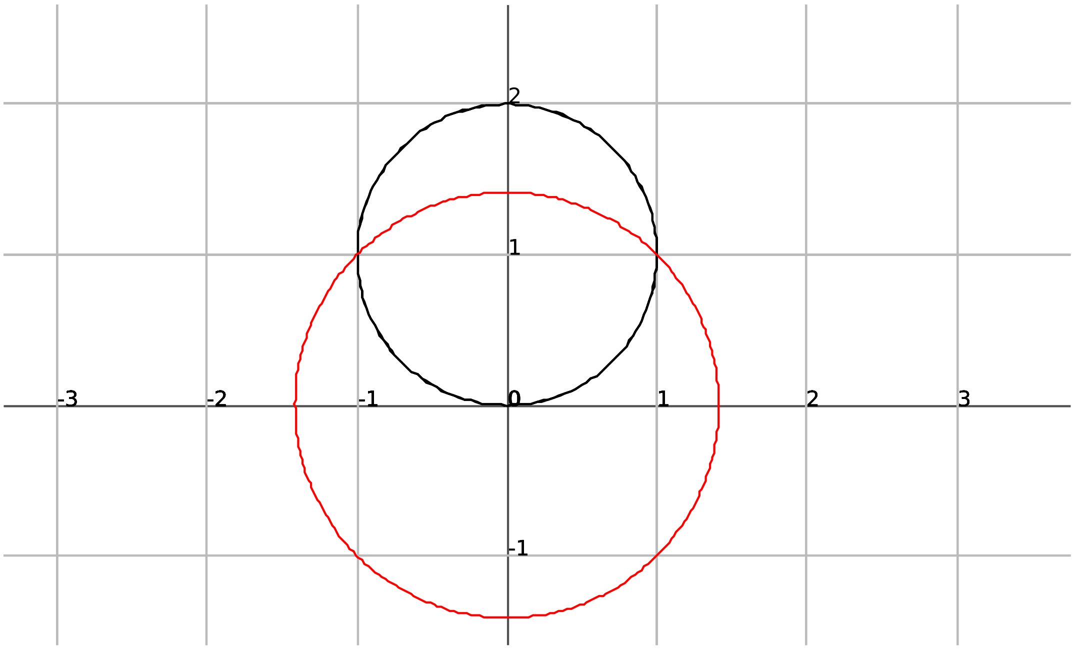
$$= \int_0^{\frac{\pi}{3}} 4 + 8\cos\theta + 4\cos^2\theta - 9\ d\theta = \int_0^{\frac{\pi}{3}} -5 + 8\cos\theta + 4\left(\frac{1+\cos(2\theta)}{2}\right)\ d\theta$$

$$= \int_0^{\frac{\pi}{3}} -5 + 8\cos\theta + 2 + 2\cos(2\theta)\ d\theta = \int_0^{\frac{\pi}{3}} -3 + 8\cos\theta + 2\cos(2\theta)\ d\theta$$

$$= -3\theta + 8\sin\theta + \sin(2\theta)\Big|_0^{\frac{\pi}{3}} = \left(-\pi + 8\sin\left(\tfrac{\pi}{3}\right) + \sin\left(\tfrac{2\pi}{3}\right)\right) - (0 + 8\sin 0 + \sin 0)$$

$$= -\pi + 8\left(\tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{\sqrt{3}}{2}\right) = -\pi + 4\sqrt{3} + \tfrac{\sqrt{3}}{2} = \boxed{\frac{9\sqrt{3}}{2} - \pi}$$

6. Find the area bounded inside $r = 2\sin\theta$ and outside $r = \sqrt{2}$.



These two polar curves intersect when $2\sin\theta = \sqrt{2} \Rightarrow \sin\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$ to $\theta = \frac{3\pi}{4}$. Using symmetry, we will integrate from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$ and double that area.

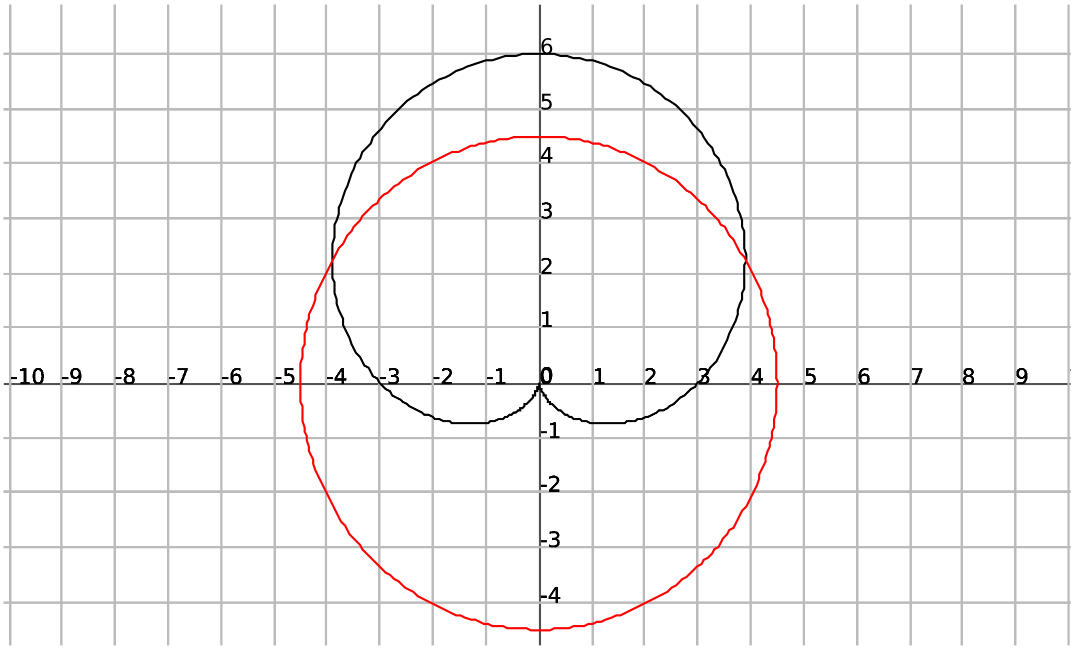
$$\text{Area} = A = 2\left(\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left((\text{outer } \ r)^2 - (\text{inner } \ r)^2\right)\ d\theta\right) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left((2\sin\theta)^2 - (\sqrt{2})^2\right)\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\sin^2\theta - 2\ d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\left(\frac{1-\cos(2\theta)}{2}\right) - 2\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2 - 2\cos(2\theta) - 2\ d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -2\cos(2\theta)\ d\theta$$

$$= -\sin(2\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\sin\pi - \left(-\sin\left(\frac{\pi}{2}\right)\right) = 0 + 1 = \boxed{1}$$

7. Find the area bounded inside $r = 3 + 3\sin\theta$ and outside $r = \frac{9}{2}$.



These two polar curves intersect when $3 + 3\sin\theta = \frac{9}{2} \Rightarrow 3\sin\theta = \frac{3}{2} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ to $\theta = \frac{5\pi}{6}$. Using symmetry, we will integrate from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{2}$ and double that area.

$$\text{Area} = A = 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left((\text{outer} \quad r)^2 - (\text{inner} \quad r^2)\right)\, d\theta\right) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left((3+3\sin\theta)^2 - \left(\frac{9}{2}\right)^2\right)\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9 + 18\sin\theta + 9\sin^2\theta - \frac{81}{4}\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9 + 18\sin\theta + 9\left(\frac{1-\cos(2\theta)}{2}\right) - \frac{81}{4}\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9 + 18\sin\theta + \frac{9}{2} - \frac{9}{2}\cos(2\theta) - \frac{81}{4}\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -\frac{27}{4} + 18\sin\theta - \frac{9}{2}\cos(2\theta)\, d\theta$$

$$= -\frac{27}{4}\theta - 18\cos\theta - \frac{9}{4}\sin(2\theta)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(-\frac{27\pi}{8} - 18\cos\left(\frac{\pi}{2}\right) - \frac{9}{4}\sin(\pi)\right) - \left(-\frac{27\pi}{24} - 18\cos\left(\frac{\pi}{6}\right) - \frac{9}{4}\sin(\frac{\pi}{3})\right)$$

$$= -\frac{27\pi}{8} - 0 - 0 + \frac{9\pi}{8} + 18\left(\frac{\sqrt{3}}{2}\right) + \frac{9}{4}\left(\frac{\sqrt{3}}{2}\right) = -\frac{27\pi}{8} - 0 - 0 + \frac{9\pi}{8} + 9\sqrt{3} + \frac{9\sqrt{3}}{8}$$

$$= \frac{81\sqrt{3}}{8} - \frac{18\pi}{8} = \boxed{\frac{81\sqrt{3}}{8} - \frac{9\pi}{4}}$$

Differential Equations: Solve each of the following differential equations, finding either the general or particular solution, as needed.

8. $\dfrac{dy}{dx} = -6xy$ with $y(0) = 7$ Separable

$\frac{1}{y} dy = -6x dx$

Antidifferentiate: $\ln|y| = -3x^2 + C \Rightarrow y = \pm e^C e^{-3x^2}$ or $y = Ke^{-3x^2}$ where $K = \pm e^C$.

Use intitial-condition: $7 = Ke^0 \Rightarrow K = 7$

Finally, $\boxed{y = 7e^{-3x^2}}$

9. $x\dfrac{dy}{dx} - y = x$ with $y(1) = 7$ Linear First Order

Linear Form: $\dfrac{dy}{dx} - \dfrac{1}{x} y = 1$

Integrating Factor: $I(x) = e^{\int P(x)\,dx} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = e^{\ln\left(|x|^{-1}\right)} = \dfrac{1}{|x|}$

Take $I(x) = \dfrac{1}{x}$.

Multiply Diff. Eq. in its linear form by $I(x)$:

$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = \frac{1}{x}$

Recognize left side as a product rule derivative:

$\left(\frac{1}{x} y\right)' = \frac{1}{x}$

Antidifferentiate:

$\frac{1}{x} y = \ln|x| + C \Rightarrow y = x(\ln|x| + C)$

Use intitial-condition: $7 = 1(\ln(1) + C) \Rightarrow C = 7$

Finally, $y = x(\ln|x| + 7)$ or $\boxed{y = 7x + x\ln|x|}$

10. $\dfrac{dy}{dx} = 6x(y-1)^{\frac{2}{3}}$ Separable

$\frac{1}{(y-1)^{\frac{2}{3}}} dy = 6x dx$

Antidifferentiate: $\displaystyle\int \frac{1}{(y-1)^{\frac{2}{3}}}\,dy = \int 6x\,dx$

u-substitution $\displaystyle\int \frac{1}{u^{\frac{2}{3}}}\,du = \int 6x\,dx$ $\boxed{\begin{array}{l} u = y - 1 \\ du = dy \end{array}}$

$3u^{\frac{1}{3}} = 3x^2 + C$

$3(y-1)^{\frac{1}{3}} = 3x^2 + C$

Finally, $(y-1)^{\frac{1}{3}} = \frac{3x^2+C}{3} = x^2 + \dfrac{C}{3} \Rightarrow (y-1) = \left(x^2 + \dfrac{C}{3}\right)^3 \Rightarrow \boxed{y = \left(x^2 + \dfrac{C}{3}\right)^3 + 1}$

or $y = (x^2 + K)^3 + 1$ where the constant $K = \dfrac{C}{3}$.

11. $\dfrac{dy}{dx} + 3y = 2xe^{-3x}$ Linear First Order

(already in) Linear Form: $\dfrac{dy}{dx} + 3y = 2xe^{-3x}$

Integrating Factor: $I(x) = e^{\int P(x)\ dx} = e^{\int 3\ dx} = e^{3x}$

Take $I(x) = e^{3x}$.

Multiply Diff. Eq. in its linear form by $I(x)$:

$e^{3x}\dfrac{dy}{dx} + 3e^{3x}y = 2xe^{-3x}e^{3x}$

$e^{3x}\dfrac{dy}{dx} + 3e^{3x}y = 2xe^{0}$

$e^{3x}\dfrac{dy}{dx} + 3e^{3x}y = 2x$

Recognize left side as a product rule derivative:

$\left(e^{3x}y\right)' = 2x$

Antidifferentiate:

$e^{3x}y = x^2 + C \Rightarrow y = \dfrac{x^2+C}{e^{3x}}$

Finally, $\boxed{y = e^{-3x}(x^2 + C)}$

12. $2x\dfrac{dy}{dx} + y = 10\sqrt{x}$ Linear First Order

Linear Form: $\dfrac{dy}{dx} + \dfrac{1}{2x}y = \dfrac{5}{\sqrt{x}}$

Integrating Factor: $I(x) = e^{\int P(x)\ dx} = e^{\int \frac{1}{2x}\ dx} = e^{\frac{1}{2}\ln|x|} = e^{\ln(|x|^{\frac{1}{2}})} = \sqrt{x}$

Take $I(x) = \sqrt{x}$

Multiply Diff. Eq. in its linear form by $I(x)$:

$\sqrt{x}\dfrac{dy}{dx} + \dfrac{\sqrt{x}}{2x}y = 5$

$\sqrt{x}\dfrac{dy}{dx} + \dfrac{1}{2\sqrt{x}}y = 5$

Recognize left side as a product rule derivative:

$(\sqrt{x}y)' = 5$

Antidifferentiate:

$\sqrt{x}y = 5x + C \Rightarrow y = \dfrac{5x + C}{\sqrt{x}}$

Finally, $\boxed{y = 5\sqrt{x} + Cx^{-\frac{1}{2}}}$

13. $\dfrac{dy}{dx} = (1 + y^2)e^x$ Separable

$\dfrac{1}{1 + y^2} dy = e^x dx$

Antidifferentiate: $\displaystyle\int \frac{1}{1 + y^2}\ dy = \int e^x\ dx$

$\arctan y = e^x + C$

Finally, $\boxed{y = \tan(e^x + C)}$

14. $\dfrac{dy}{dx} + 2xy = x$ with $y(0) = -2$ Linear First Order

(already in) Linear Form: $\dfrac{dy}{dx} + 2xy = x$

Integrating Factor: $I(x) = e^{\int P(x)\ dx} = e^{\int 2x\ dx} = e^{x^2}$

Take $I(x) = e^{x^2}$

Multiply Diff. Eq. in its linear form by $I(x)$:

$e^{x^2}\dfrac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$

Recognize left side as a product rule derivative:

$\left(e^{x^2}y\right)' = xe^{x^2}$

Antidifferentiate:

$e^{x^2}y = \displaystyle\int xe^{x^2}\ dx = \frac{1}{2}e^{x^2} + C$ using u-substitution.

$\Rightarrow y = \dfrac{\frac{1}{2}e^{x^2} + C}{e^{x^2}} \Rightarrow y = \dfrac{1}{2} + Ce^{-x^2}$

Use intitial-condition: $-2 = \dfrac{1}{2} + Ce^0 \Rightarrow C = -\dfrac{5}{2}$

Finally, $\boxed{y = \dfrac{1}{2} - \dfrac{5}{2}e^{-x^2}}$

15. $\dfrac{dy}{dx} = \ln x\sqrt{1-y^2}$ Separable

$$\frac{1}{\sqrt{1-y^2}}\,dy = \ln x\,dx$$

Antidifferentiate: $\displaystyle\int \frac{1}{\sqrt{1-y^2}}\,dy = \int \ln x\,dx = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - \int 1\,dx$

$$\boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \dfrac{1}{x}\,dx & v = x \end{array}}$$

$\arcsin y = x\ln x - x + C$ by Integration by Parts

Finally, $\boxed{y = \sin(x\ln x - x + C)}$

16. $\dfrac{dy}{dx} = \dfrac{x\arctan x}{y}$ Separable

$$y\,dy = x\arctan x\,dx$$

Antidifferentiate: $\displaystyle\int y\,dy = \int x\arctan x\,dx$

Integration by Parts:

$$\boxed{\begin{array}{ll} u = \arctan x & dv = x\,dx \\ du = \dfrac{1}{1+x^2}\,dx & v = \dfrac{x^2}{2} \end{array}}$$

$$\frac{y^2}{2} = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2}\arctan x - \frac{1}{2}\int 1 - \frac{1}{x^2+1}\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2}\arctan x - \frac{1}{2}\left(x - \arctan x + C\right)$$

$$y^2 = x^2\arctan x - x + \arctan x - C$$

Finally, $\boxed{y = \pm\sqrt{x^2\arctan x - x + \arctan x - C}}$

17. $x^3\dfrac{dy}{dx} + x^2y = 2x^3 + 1$ Linear First Order

Linear Form: $\dfrac{dy}{dx} + \dfrac{1}{x}y = 2 + \dfrac{1}{x^3}$

Integrating Factor: $I(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln|x|} = |x|$

Take $I(x) = x$

Multiply Diff. Eq. in its linear form by $I(x)$:

$x\dfrac{dy}{dx} + y = 2x + \dfrac{1}{x^2}$

Recognize left side as a product rule derivative:

$(xy)' = 2x + \dfrac{1}{x^2}$

Antidifferentiate:

$xy = \displaystyle\int 2x + x^{-2}\,dx = x^2 - x^{-1} + C$

Finally, $\boxed{y = x - \dfrac{1}{x^2} + \dfrac{C}{x}}$

18. $(1+x)\dfrac{dy}{dx} + y = \cos x$ with $y(0) = 1$ Linear First Order

Linear Form: $\dfrac{dy}{dx} + \dfrac{1}{1+x}y = \dfrac{\cos x}{1+x}$

Integrating Factor: $I(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{1+x}\,dx} = e^{\ln|1+x|} = |1+x|$

Take $I(x) = 1 + x$

Multiply Diff. Eq. in its linear form by $I(x)$:

$(1+x)\dfrac{dy}{dx} + y = \cos x$

Note this puts it back into the original form. Technically we didn't need the integrating factor here, but it's ok if you don't see that right away. The integrating factor still gets you into the right form regardless.

Recognize left side as a product rule derivative:

$((1+x)y)' = \cos x$

Antidifferentiate:

$(1+x)y = \displaystyle\int \cos x\,dx = \sin x + C \Rightarrow y = \dfrac{\sin x + C}{1+x}$

Use intitial-condition: $1 = \dfrac{\sin 0 + C}{1+0} \Rightarrow C = 1$

Finally, $\boxed{y = \dfrac{\sin x + 1}{1+x}}$

19. $\dfrac{dy}{dx} = 3(y+7)x^2$ is both linear and separable.

Solve it both ways and compare your answers.

First, Separable

$$\frac{1}{y+7}\ dy = 3x^2\ dx$$

Antidifferentiate: $\displaystyle\int \frac{1}{y+7}\ dy = \int 3x^2\ dx$

$\Rightarrow \ln|y+7| = x^3 + C \Rightarrow |y+7| = e^{x^3+C} = e^{x^3}e^C \Rightarrow y+7 = \pm e^C e^{x^3}$

Finally, $\boxed{y = -7 + Ke^{x^3}}$ where the constant $K = \pm e^C$

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Second, Linear First Order

$$\frac{dy}{dx} = 3(y+7)x^2$$

$$\frac{dy}{dx} = 3x^2y + 21x^2$$

Linear Form: $\displaystyle\frac{dy}{dx} - 3x^2y = 21x^2$

Integrating Factor: $I(x) = e^{\int P(x)\ dx} = e^{\int -3x^2\ dx} = e^{-x^3}$

Take $I(x) = e^{-x^3}$

Multiply Diff. Eq. in its linear form by $I(x)$:

$$e^{-x^3}\frac{dy}{dx} - 3x^2e^{-x^3}y = 21x^2e^{-x^3}$$

Recognize left side as a product rule derivative:

$$\left(e^{-x^3}y\right)' = 21x^2e^{-x^3}$$

Antidifferentiate:

$$e^{-x^3}y = 21\int x^2e^{-x^3}\ dx = -\frac{21}{3}\int e^u\ du = -7e^u + C = -7e^{-x^3} + C$$

$$\boxed{\begin{array}{l} u = -x^3 \\ du = -3x^2\ dx \\ -\frac{1}{3}\ du = x^2\ dx \end{array}}$$

Then $e^{-x^3}y = -7e^{-x^3} + C \Rightarrow y = \dfrac{-7e^{-x^3} + C}{e^{-x^3}}$

Finally, $\boxed{y = -7 + Ce^{x^3}}$ Match!