Math 12 Midterm Exam #3 April 27, 2011

Answer Key

1. [15 Points] Find the **Interval** and **Radius** of Convergence for each of the following power series. Analyze carefully and with full justification.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!\; x^{2n+1}}{(3n)!}$

Use Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\dfrac{(2(n+1))!\; x^{2(n+1)+1}}{(3(n+1))!}}{\dfrac{(2n)!\; x^{2n+1}}{(3n)!}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \right| \cdot \frac{(2n+2)!}{(2n)!} \cdot \frac{(3n)!}{(3n+3)!}$$

$$= \lim_{n\to\infty} |x^2| \cdot \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \cdot \frac{(3n)!}{(3n+3)(3n+2)(3n+1)(3n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)|x|^2}{(3n+3)(3n+2)(3n+1)} = 0 < 1 \text{ for all } x.$$

The Ratio Test gives convergence for all x. So Interval of Convergence $\boxed{I = (-\infty, \infty)}$ with Radius of Convergence $\boxed{R = \infty}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x+5)^n}{n^2 7^n}$

Use Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\dfrac{(3x+5)^{n+1}}{(n+1)^2 7^{n+1}}}{\dfrac{(3x+5)^n}{n^2 7^n}} \right| = \lim_{n\to\infty} \left| \frac{(3x+5)^{n+1}}{(3x+5)^n} \right| \cdot \frac{n^2}{(n+1)^2} \cdot \frac{7^n}{7^{n+1}} = \frac{|3x+5|}{7}$$

The Ratio Test gives convergence for x when $\dfrac{|3x+5|}{7} < 1$ or $|3x+5| < 7$.

That is $-7 < 3x+5 < 7 \Longrightarrow -12 < 3x < 2 \Longrightarrow -4 < x < \dfrac{2}{3}$

Endpoints:

• $x = \dfrac{2}{3}$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{7^n}{n^2 7^n} = \sum_{n=0}^{\infty} \frac{1}{n^2}$ which is a convergent p-series with $p = 2 > 1$.

• $x = -4$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-7)^n}{n^2 7^n} \sum_{n=0}^{\infty} \frac{(-1)^n 7^n}{n^2 7^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ which is convergent by Absolute Convergence Test since the absolute series above converges. (can also use AST here, but not needed if you use ACT)

Finally, Interval of Convergence $\boxed{I = \left[-4, \frac{2}{3}\right]}$ with Radius of Convergence $\boxed{R = \frac{7}{3}}$.

2. [10 Points] Consider the function $f(x)$ that satisfies the following

$$f(4) = 2$$

$$f'(4) = -3$$

$$f''(4) = \frac{6}{7}$$

$$f'''(4) = -1$$

Find the **Taylor polynomial of degree 3** for $\mathbf{f(x)}$ centered at $\mathbf{a = 4}$.

$$T_3(x) = f(4) + f'(4)(x-2) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

$$= 2 - 3(x-4) + \frac{\frac{6}{7}}{2!}(x-4)^2 - \frac{1}{3!}(x-4)^3$$

$$= \boxed{2 - 3(x-4) + \frac{3}{7}(x-4)^2 - \frac{1}{6}(x-4)^3}$$

3. [10 Points] Find the **MacLaurin series** representation for each of the following functions. State the Radius of Convergence for each series. Your answer should be in sigma notation $\sum_{n=0}^{\infty}$

(a) $f(x) = x^2 e^{-3x}$

First $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Then $e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!}$

Finally, $x^2 e^{-3x} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{n+2}}{n!}}$

(b) $f(x) = x\arctan(3x)$

First $\arctan x = \int \dfrac{1}{1+x^2}\ dx = \displaystyle\int \frac{1}{1-(-x^2)}\ dx = \int \sum_{n=0}^{\infty} \left(-x^2\right)^n\ dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\ dx$

$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. Here $R = 1$ or $|x| < 1$.

Next, $\arctan(3x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}x^{2n+1}}{2n+1}$

Finally, $x\arctan(3x) = x\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}x^{2n+2}}{2n+1}$. Here $|3x| < 1$ so $|x| < \frac{1}{3}$, or $R = \frac{1}{3}$.

4. [10 Points] Use a Power Series representation for $x\ln(1+x^3)$ to estimate the given integral within the given error. Justify in words that your error is indeed less than $\dfrac{1}{10}$.

$$\text{Estimate} \quad \int_0^1 x\ln(1+x^3)\ dx \quad \text{with error less than } \frac{1}{10}$$

First $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \displaystyle\sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$.

Then, $\ln(1+x) = \int \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$.

Next, $\ln(1+x^3) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n+1}$.

Finally, $x\ln(1+x^3) = x\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+4}}{n+1}$

Now, $\displaystyle\int_0^1 x\ln(1+x^3)\ dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+4}}{n+1}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+5}}{(3n+5)(n+1)}\Bigg|_0^1$

$= \displaystyle\frac{x^5}{5} - \frac{x^8}{8\cdot 2} + \frac{x^{11}}{11\cdot 3} + \ldots\Bigg|_0^1 = \left(\frac{1}{5} - \frac{1}{16} + \frac{1}{33} + \ldots\right) - (0 - 0 + 0 + \ldots) \approx \boxed{\frac{1}{5}}$

Note this is an alternating series. Use the Alternating Series Estimation Theorem. If we approximate the actual sum with only the first term, the error from the actual sum will be at most the absolute value of the next term, $\dfrac{1}{16}$. Here $\dfrac{1}{16} < \dfrac{1}{10}$ as desired.

5. [15 Points] Find the **sum** for each of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 7^n}{9^n n!} = \sum_{n=0}^{\infty} \frac{\left(-\frac{7}{9}\right)^n}{n!} = \boxed{e^{-\frac{7}{9}}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(3^2)^n(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n+1)!}$

$\displaystyle= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{\left(\frac{\pi}{3}\right)(2n+1)!} = \left(\frac{3}{\pi}\right) \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \left(\frac{3}{\pi}\right) \sin\left(\frac{\pi}{3}\right) = \left(\frac{3}{\pi}\right)\frac{\sqrt{3}}{2} = \boxed{\frac{3\sqrt{3}}{2\pi}}$

6. [20 Points] Volumes of Revolution

(a) Consider the region bounded by $y = \cos x$, $y = e^x + 2$, $x = 0$, and $x = \frac{\pi}{2}$. Rotate this region about the horizontal line $\boxed{y = -3}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Washer Method. Sketch the solid, along with one of the approximating washers.

See me for a sketch.

$$V = \int_0^{\frac{\pi}{2}} \pi \left((\text{outer radius})^2 - (\text{inner radius})^2\right)\, dx = \boxed{\pi \int_0^{\frac{\pi}{2}} (e^x + 5)^2 - (\cos x + 3)^2\, dx}$$

(b) Consider the region bounded by $y = e^x$, $y = \ln x$, $x = 1$ and $x = 2$. Rotate the region about the y-axis. **COMPUTE** the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

See me for a sketch.

$V = \displaystyle\int_1^2 2\pi \text{ radius height } dx = \int_1^2 2\pi x(e^x - \ln x)\, dx = 2\pi \left(\int_1^2 xe^x\, dx - \int_1^2 x \ln x\, dx\right)$ I.B.P. here for both pieces

$$= 2\pi \left(xe^x - e^x \Big|_1^2 - \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_1^2 \right) = 2\pi \left((2e^2 - e^2 - (e^1 - e^1)) - \left(\frac{4}{2}\ln 2 - \frac{4}{4} - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right)\right)\right)$$

$$= 2\pi \left(e^2 - 2\ln 2 + 1 - \frac{1}{4}\right) = \boxed{2\pi \left(e^2 - \ln 4 + \frac{3}{4}\right)}$$

7. [20 Points] Consider the Parametric Curve represented by $x = 3 - 2t$ and $y = e^t + e^{-t}$.

(a) **Compute** the **arclength** of this parametric curve for $0 \le t \le 1$.

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(-2)^2 + (e^t - e^{-t})^2}\, dt = \int_0^1 \sqrt{4 + e^{2t} - 2 + e^{-2t}}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt = \int_0^1 \sqrt{(e^t + e^{-t})^2}\, dt$$

$$= \int_0^1 e^t + e^{-t}\, dt = e^t - e^{-t}\Big|_0^1 = \left(e - e^{-1}\right) - \left(e^0 - e^0\right) = e - e^{-1} = \boxed{e - \frac{1}{e}}$$

(b) **Compute** the **surface area** obtained by rotating this curve about the x-**axis**, for $0 \le t \le 1$.

$$\text{S.A.} = \int_0^1 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi y(t) \sqrt{(-2)^2 + (e^t - e^{-t})^2}\, dt$$

$\displaystyle\int_0^1 2\pi \left(e^t + e^{-t}\right) \sqrt{(e^t + e^{-t})^2}\, dt = \int_0^1 2\pi \left(e^t + e^{-t}\right) \left((e^t + e^{-t}\right)\, dt$ from part(b) above.

$$= 2\pi \int_0^1 e^{2t}+2+e^{-2t}\, dt = 2\pi \left(\frac{1}{2}e^{2t} + 2t - \frac{1}{2}e^{-2t}\right)\Big|_0^1 = 2\pi \left(\left(\frac{1}{2}e^2 + 2 - \frac{1}{2}e^{-2}\right) - \left(\frac{1}{2}e^0 + 0 - \frac{1}{2}e^0\right)\right)$$

$$= 2\pi \left(\left(\tfrac{1}{2}e^2 + 2 - \tfrac{1}{2}e^{-2}\right) - \left(\tfrac{1}{2} + 0 - \tfrac{1}{2}\right)\right) = 2\pi \left(\tfrac{1}{2}e^2 + 2 - \tfrac{1}{2}e^{-2}\right) = \boxed{\pi \left(e^2 + 4 - e^{-2}\right)}$$

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute the **sum** $\displaystyle\sum_{n=0}^{\infty} \frac{n}{7^n}$

First $\displaystyle\sum_{n=0}^{\infty} \frac{n}{7^n}$ comes from $\displaystyle\sum_{n=0}^{\infty} nx^n$ with $x = \dfrac{1}{7}$.

Recognize $\displaystyle\sum_{n=0}^{\infty} nx^n = \sum_{n=0}^{\infty} nx^{n-1}x = x\sum_{n=0}^{\infty} nx^{n-1} = x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = x\frac{d}{dx}\left(\frac{1}{1-x}\right)$

$$= x\left(\frac{1}{(1-x)^2}\right) = \frac{x}{(1-x)^2}$$

Finally, substituting $x = \dfrac{1}{7}$, we get

$$\sum_{n=0}^{\infty} \frac{n}{7^n} = \frac{\frac{1}{7}}{\left(1 - \frac{1}{7}\right)^2} = \frac{\frac{1}{7}}{\left(\frac{6}{7}\right)^2} = \frac{\frac{1}{7}}{\frac{36}{49}} = \boxed{\frac{7}{36}}$$

OPTIONAL BONUS #2 Compute the **sum** $\sum_{n=0}^{\infty} \frac{n^2}{7^n n!}$

First $\sum_{n=0}^{\infty} \frac{n^2}{7^n n!}$ comes from $\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!}$ with $x = \frac{1}{7}$

Recognize $\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!} = \sum_{n=0}^{\infty} \frac{n^2 x^{n-1} x}{n!} = x \sum_{n=0}^{\infty} \frac{n \cdot n x^{n-1}}{n!} = x \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{n x^n}{n!} \right) = x \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{n x^{n-1} x}{n!} \right)$

$= x \frac{d}{dx} \left(x \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} \right) = x \frac{d}{dx} \left(x \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \right) = x \frac{d}{dx} \left(x \frac{d}{dx} e^x \right)$

$= x \frac{d}{dx} (x e^x) = x (x e^x + e^x) = x^2 e^x + x e^x$

Finally, substituting $x = \frac{1}{7}$, we get

$$\sum_{n=0}^{\infty} \frac{n^2}{7^n n!} = \left(\left(\frac{1}{7} \right)^2 + \frac{1}{7} \right) e^{\frac{1}{7}} = \left(\frac{1}{49} + \frac{1}{7} \right) e^{\frac{1}{7}} = \boxed{\frac{8}{49} e^{\frac{1}{7}}}$$

OPTIONAL BONUS #3 Compute the sum $\sum_{n=0}^{\infty} \frac{n^3}{7^n}$

First $\sum_{n=0}^{\infty} \frac{n^3}{3^n}$ comes from $\sum_{n=0}^{\infty} n^3 x^n$ with $x = \frac{1}{7}$

Recognize $\sum_{n=0}^{\infty} n^3 x^n = \sum_{n=0}^{\infty} n^3 x^{n-1} x = x \sum_{n=0}^{\infty} n^2 \cdot n x^{n-1} = x \frac{d}{dx} \left(\sum_{n=0}^{\infty} n^2 x^n \right)$

$= x \frac{d}{dx} \left(\sum_{n=0}^{\infty} n \cdot n x^{n-1} x \right) = x \frac{d}{dx} \left(x \sum_{n=0}^{\infty} n \cdot n x^{n-1} \right) = x \frac{d}{dx} \left(x \frac{d}{dx} \left(\sum_{n=0}^{\infty} n x^n \right) \right)$

$= x \frac{d}{dx} \left(x \frac{d}{dx} \left(\sum_{n=0}^{\infty} n x^{n-1} x \right) \right) = x \frac{d}{dx} \left(x \frac{d}{dx} \left(x \sum_{n=0}^{\infty} n x^{n-1} \right) \right)$

$= x \frac{d}{dx} \left(x \frac{d}{dx} \left(x \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) \right) \right) = x \frac{d}{dx} \left(x \frac{d}{dx} \left(x \frac{d}{dx} \left(\frac{1}{1-x} \right) \right) \right)$

$= x \frac{d}{dx} \left(x \frac{d}{dx} \left(x \left(\frac{1}{(1-x)^2} \right) \right) \right) = x \frac{d}{dx} \left(x \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) \right)$

$= x \frac{d}{dx} \left(x \left(\frac{(1-x)^2(1) - x(2(1-x)(-1))}{(1-x)^4} \right) \right) = x \frac{d}{dx} \left(x \left(\frac{(1-x)((1-x) + 2x)}{(1-x)^4} \right) \right)$

$= x \frac{d}{dx} \left(x \left(\frac{1+x}{(1-x)^3} \right) \right) = x \frac{d}{dx} \left(\frac{x + x^2}{(1-x)^3} \right) = x \left(\frac{(1-x)^3(1+2x) - (x+x^2)3(1-x)^2(-1)}{(1-x)^6} \right)$

$= x \left(\frac{(1-x)^2((1-x)(1+2x) + 3(x+x^2))}{(1-x)^6} \right) = x \left(\frac{(1-x)(1+2x) + 3(x+x^2)}{(1-x)^4} \right)$

$$= x\left(\frac{1 - x + 2x - 2x^2 + 3x + 3x^2}{(1-x)^4}\right) = x\left(\frac{1 + 4x + x^2}{(1-x)^4}\right) = \frac{x + 4x^2 + x^3}{(1-x)^4}$$

Finally, substituting $x = \frac{1}{7}$, we get $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{7^n} = \frac{\frac{1}{7} + 4\left(\frac{1}{7}\right)^2 + \left(\frac{1}{7}\right)^3}{\left(1 - \frac{1}{7}\right)^4} = \boxed{\frac{91}{216}}$