Math 12 Midterm Exam #2 March 30, 2011

Answer Key

1. [40 Points] Compute the following integral, or else show that it diverges.

(a) $\int_1^9 \frac{1}{(x-7)^2}\,dx = \int_1^7 \frac{1}{(x-7)^2}\,dx + \int_7^9 \frac{1}{(x-7)^2}\,dx = \lim_{s\to 7^-} \int_1^s \frac{1}{(x-7)^2}\,dx + \lim_{t\to 7^+} \int_t^9 \frac{1}{(x-7)^2}\,dx$

$= \lim_{s\to 7^-} -\frac{1}{x-7}\Big|_1^s + \lim_{t\to 7^+} -\frac{1}{x-7}\Big|_t^9 = \lim_{s\to 7^-} -\frac{1}{s-7} - \left(-\frac{1}{1-7}\right) + \lim_{t\to 7^+} -\frac{1}{9-7} - \left(-\frac{1}{t-7}\right)$

$= \lim_{s\to 7^-} -\frac{1}{s-7} - \frac{1}{6} + \lim_{t\to 7^+} -\frac{1}{2} + \frac{1}{t-7} = -\frac{1}{0^-} - \frac{1}{6} - \frac{1}{2} + \frac{1}{0^+} = \infty + \infty$

The integral $\boxed{\text{diverges}}$ since either of the two split integrals diverge.

Note: you can also do a u-substitution changing your limits of integration.

(b) $\int \frac{4x+1}{x^2-3x-10}\,dx = \int \frac{4x+1}{(x-5)(x+2)}\,dx = \int \frac{3}{x-5} + \frac{1}{x+2}\,dx = \boxed{3\ln|x-5| + \ln|x+2| + C}$

Partial Fractions Decomposition:

$\frac{4x+1}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}$

Clearing the denominator yields:

$4x+1 = A(x+2) + B(x-5)$
$4x+1 = (A+B)x + (2A-5B)$
so that $A+B=4$ and $2A-5B=1$
Solve for $A=3$ and $B=1$

Note: Once you find the partial fractions decomposition (find A and B) you can find the common denominator and check that your decomposition is equal to the original integrand.

(c) $\int \frac{x^4+x^2+x+1}{x^3+x}\,dx = \int x + \frac{x+1}{x^3+x}\,dx = \int x + \frac{x+1}{x(x^2+1)}\,dx = \int x + \frac{1}{x} + \frac{-x+1}{x^2+1}\,dx =$

$\int x + \frac{1}{x} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx = \boxed{\frac{x^2}{2} + \ln|x| - \frac{\ln|x^2+1|}{2} + \arctan x + C}$

Long division yields:

$$\begin{array}{r} x \\ x^3+x\,\overline{)\,x^4+x^2+x+1} \\ \underline{-(x^4+x^2)} \\ x+1 \end{array}$$

Partial Fractions Decomposition:

$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

Clearing the denominator yields:

$x+1 = A(x^2+1)+(Bx+C)x$
$x+1 = (A+B)x^2+Cx+A$
so that $A+B=0,\ C=1$ and $A=1$
Solve for $A=1,\ C=1$ and $B=-1$

(d) $$\int_9^{\infty} \frac{1}{x^2-8x+41}\,dx = \lim_{t\to\infty}\int_9^t \frac{1}{x^2-8x+41}\,dx = \lim_{t\to\infty}\int_9^t \frac{1}{(x-4)^2+25}\,dx \quad \text{complete the square}$$

Substitute

$u = x-4$	$x=9 \Rightarrow u=5$
$du = dx$	$x=t \Rightarrow u=t-4$

$$= \lim_{t\to\infty}\int_5^{t-4} \frac{1}{u^2+25}\,du = \lim_{t\to\infty}\frac{1}{25}\int_5^{t-4}\frac{1}{\left(\frac{u}{5}\right)^2+1}\,du = \lim_{t\to\infty}\frac{5}{25}\int_1^{\frac{t-4}{5}}\frac{1}{v^2+1}\,dv$$

Substitute

$v=\frac{u}{5}$	$u=5\Rightarrow v=1$
$dv=\frac{1}{5}du$	$u=t-4\Rightarrow v=\frac{t-4}{5}$
$5dv=du$	

$$= \lim_{t\to\infty}\frac{1}{5}\arctan v\Big|_1^{\frac{t-4}{5}} = \lim_{t\to\infty}\frac{1}{5}\left(\arctan\left(\frac{t-4}{5}\right)-\arctan 1\right) = \frac{1}{5}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{1}{5}\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{20}}$$

OR you could skip all the substitution steps and go straight to

$$\lim_{t\to\infty}\int_9^t \frac{1}{(x-4)^2+25}\,dx = \lim_{t\to\infty}\frac{1}{5}\left(\arctan\left(\frac{x-4}{5}\right)\right)\Big|_9^t = \ldots$$

using the formula $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$

2. [8 Points] Determine whether the following sequence **converges** or **diverges**. If it converges, compute its limit. Justify your answer. Do not just put down a number.

$$\left\{\left(\frac{n+1}{n}\right)^n\right\}_{n=1}^{\infty}$$

We switch to the variable x and the related function $f(x)=\left(\frac{x+1}{x}\right)^x$ in order to apply L'H Rule:

$$\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x^{1^{\infty}}} = \lim_{x\to\infty} e^{\ln\left(\frac{x+1}{x}\right)^x} = e^{\lim_{x\to\infty}\ln\left(\frac{x+1}{x}\right)^x} = e^{\lim_{x\to\infty} x\ln\left(\frac{x+1}{x}\right)^{\infty\cdot 0}}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}}} \overset{\frac{0}{0}}{\underset{}{}} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\left(\frac{x}{x+1}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty}\left(\frac{x}{x+1}\right)} = e^1 = \boxed{e}$$

Finally, $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$, so that the sequence $\left\{\left(\frac{n+1}{n}\right)^n\right\}_{n=1}^{\infty}$ converges to e.

3. [8 Points] Find the **sum** of the following series (which does converge):

$$\sum_{n=1}^{\infty} \frac{(-1)^n\ 3^{n+2}}{2^{4n-1}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n\ 3^{n+2}}{2^{4n-1}} = -\frac{3^3}{2^3} + \frac{3^4}{2^7} - \frac{3^5}{2^{11}} + \dots$$

Here we have a nice geometric series with $a = -\frac{27}{8}$ and $r = -\frac{3}{2^4} = -\frac{3}{16}$

As a result, the sum is given by $\frac{a}{1-r} = \frac{-\frac{27}{8}}{1-\left(-\frac{3}{16}\right)} = \frac{-\frac{27}{8}}{\frac{19}{16}} = -\frac{27}{8}\cdot\frac{16}{19} = \boxed{-\frac{54}{19}}$

4. [20 Points] Determine whether each of the following series **converges** or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\sum_{n=1}^{\infty} \frac{n\ \sin^2 n}{8n^2\sqrt{n}+n+7}$

Since the terms are bounded $\frac{n\ \sin^2 n}{8n^2\sqrt{n}+n+7} \leq \frac{n}{8n^2\sqrt{n}+n+7} \leq \frac{n}{8n^{\frac{5}{2}}} = \frac{1}{8n^{\frac{3}{2}}} \leq \frac{1}{n^{\frac{3}{2}}}$ and $\sum_{0}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a convergent p-series with $p = \frac{3}{2} > 1$. Then the (smaller) original series is also convergent by CT.

(b) $\sum_{n=1}^{\infty} e^{\frac{\sin n}{n}}$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$

Since $\lim_{n\to\infty} e^{\frac{\sin n}{n}} = e^{\lim_{n\to\infty}\frac{\sin n}{n}} = e^0 = 1$ because $\lim_{n\to\infty}\frac{\sin n}{n} = 0$

Note: $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ and the two extreme terms shoot to 0 as n blows up to infinity. Hence the term in the middle is forced to also shoot to zero by the Squeeze Law.

(c) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Since the terms $\frac{1}{\ln n} \geq \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent p-series ($p = 1$), then the (larger) series also diverges by CT.

(d) $\sum_{n=1}^{\infty} \frac{n^n}{e^{2n} n!}$

Try Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{e^{2(n+1)}(n+1)!}}{\frac{n^n}{e^{2n} n!}} \right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{e^{2n}}{e^{2n+2}} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{n^n} \cdot \frac{1}{e^2} \cdot \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \cdot \frac{1}{e^2} = e \cdot \left(\frac{1}{e^2} \right) = \frac{1}{e} < 1$$

The series is $\boxed{\text{Absolutely Convergent by the Ratio Test}}$.

5. [24 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$

First, we show the absolute series is divergent. Note that $\sum_{n=1}^{\infty} \frac{n}{n^2+1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent p-series with $p = 1$. Next,

Check: $\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \frac{n^2}{n^2+1} = 1$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is also divergent by Limit Comparison Test. As a result, we have no chance for Absolute Convergence.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

- $b_n = \frac{n}{n^2+1} > 0$

- $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

- $\frac{1}{b_{n+1}} < \frac{1}{b_n}$

  because we can show the derivative of the related function is negative, hence the terms are decreasing

Consider $f(x) = \dfrac{x}{x^2+1}$ with $f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{-x^2+1}{(x^2+1)^2} < 0$ when $-x^2+1 < 0$ or when $x > 1$ which is fine after our first term of the series.

Therefore, the original series converges by the Alternating Series Test. Finally, we can conclude the original series is $\boxed{\text{Conditionally Convergent}}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{n^2}{7^n}$

Try Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+2}(n+1)^2}{7^{n+1}}}{\dfrac{(-1)^{n+1}n^2}{7^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\cdot\frac{7^n}{7^{n+1}} = \frac{1}{7} < 1$$

The series is $\boxed{\text{Absolutely Convergent by the Ratio Test}}$.

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\,\frac{n^6+5n^2+826}{n^9+7n^3+2011}$

First, we show the absolute series is convergent using LCT. Note that $\displaystyle\sum_{n=1}^{\infty}\frac{n^6+5n^2+826}{n^9+7n^3+2011} \approx \sum_{n=1}^{\infty}\frac{1}{n^3}$ which is a convergent p-series with $p = 3 > 1$.

Check: $\displaystyle\lim_{n\to\infty}\frac{\dfrac{n^6+5n^2+826}{n^9+7n^3+2011}}{\dfrac{1}{n^3}} = \lim_{n\to\infty}\frac{n^9+5n^5+826n^3}{n^9+7n^3+2011} = 1$, which is finite and non-zero.

Therefore, these two series share the same behavior and the absolute series is also convergent by LCT. Finally the original series is convergent by the Absolute Convergence Test (ACT).

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\,\frac{1}{3^n}$ $\boxed{\text{Converges Absolutely by Absolute Convergence Test}}$ since the absolute series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n}$ is a convergent Geometric series with $|r| = \dfrac{1}{3} < 1$.

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute the sum of the following series:

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+3n} = \sum_{n=1}^{\infty} \frac{\frac{1}{3}}{n} + \frac{-\frac{1}{3}}{n+3} = \sum_{n=1}^{\infty} \frac{1}{3n} - \frac{1}{3(n+3)} = \frac{1}{3}\left(\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}\right)$

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Partial Fractions Decomposition:

$\frac{1}{n^2+3n} = \frac{A}{n} + \frac{B}{n+3}$

Clearing the denominator yields:

$1 = A(n+3) + Bn$

$1 = (A+B)n + 3A$

so that $A + B = 0$ and $3A = 1$

Solve for $A = \frac{1}{3}$ and $B = -\frac{1}{3}$

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$= \frac{1}{3}\left(1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + \frac{1}{5} - \frac{1}{8} + \frac{1}{6} - \frac{1}{9} + \frac{1}{7} - \frac{1}{10} + \dots\right)$ pieces start to cancel in this telescoping series

The n^{th} partial sum $S_n = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+3}\right)$

Finally, the original sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+3}\right) = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{1}{3}\left(\frac{11}{6}\right) = \boxed{\frac{11}{18}}$$

OPTIONAL BONUS #2

Determine whether the following series converges or diverges.

2. $\sum_{n=1}^{\infty} \frac{(-1)^n n^{4n}}{n^7 (n!)^2 e^{8n} (2n)!}$

Try Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)^{4(n+1)}}{(n+1)^7[(n+1)!]^2 e^{8(n+1)}[2(n+1)]!}}{\frac{(-1)^n n^{4n}}{n^7(n!)^2 e^{8n}(2n)!}} \right|$$

$$\lim_{n\to\infty} \frac{(n+1)^{4n+4}}{n^{4n}} \cdot \frac{n^7}{(n+1)^7} \cdot \frac{(n!)^2}{[(n+1)!]^2} \cdot \frac{e^{8n}}{e^{8n+8}} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^4(n+1)^{4n}}{n^{4n}} \cdot \frac{n^7}{(n+1)^7} \cdot \frac{n!\ n!}{(n+1)n!(n+1)n!} \cdot \frac{e^{8n}}{e^{8n}e^8} \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!}$$

$$= \lim_{n\to\infty} (n+1)^4 \left(\frac{n+1}{n}\right)^{4n} \cdot \frac{n^7}{(n+1)^7} \cdot \frac{1}{(n+1)^2} \cdot \frac{1}{e^8} \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} (n+1)^2 \left(\left(\frac{n+1}{n}\right)^n\right)^4 \cdot \frac{n^7}{(n+1)^7} \cdot \frac{1}{e^8} \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} (n+1)^2 e^4 \cdot 1 \cdot \frac{1}{e^8} \cdot \frac{1}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{(n+1)^2}{e^4(2n+2)(2n+1)} = \frac{1}{4e^4} < 1.$$

The series is absolutely convergent (and therefore convergent) by the Ratio Test .

OPTIONAL BONUS #3 Compute the following integral:

3. $\displaystyle\int \frac{\cos x}{\sin^3 x - 1}\, dx = \int \frac{1}{u^3 - 1}\, du = \int \frac{1}{(u-1)(u^2+u+1)}\, du$

Substitute $\boxed{\begin{array}{l} u = \sin x \\ \\ du = \cos x dx \end{array}}$

Partial Fractions Decomposition:

$$\frac{1}{(u-1)(u^2+u+1)} = \frac{A}{u-1} + \frac{Bu+C}{u^2+u+1}$$

Clearing the denominator yields:

$1 = A(u^2+u+1) + (Bu+C)(u-1)$
$1 = Au^2 + Au + A + Bu^2 + Cu - Bu - C$
$1 = (A+B)u^2 + (A-B+C)u + A - C$
so that $A+B=0,\ A-B+C=0$ and $A-C=1$
Solve for $A = \frac{1}{3},\ C = -\frac{2}{3}$ and $B = -\frac{1}{3}$

$$= \int \frac{1}{(u-1)(u^2+u+1)}\, du = \int \frac{\frac{1}{3}}{u-1} + \frac{-\frac{1}{3}u - \frac{2}{3}}{u^2+u+1}\, du = \frac{\ln|u-1|}{3} - \frac{1}{3}\int \frac{u+2}{u^2+u+1}\, du$$

Complete the square on the quadratic irreducible piece.

$$= \frac{\ln|u-1|}{3} - \frac{1}{3}\int \frac{u+2}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}\, du = \frac{\ln|u-1|}{3} - \frac{1}{3}\int \frac{(w-\frac{1}{2})+2}{w^2+\frac{3}{4}}\, dw = \frac{\ln|u-1|}{3} - \frac{1}{3}\int \frac{w+\frac{3}{2}}{w^2+\frac{3}{4}}\, dw$$

Substitute $\boxed{\begin{array}{l} w = u + \frac{1}{2} \Rightarrow u = w - \frac{1}{2} \\ \\ dw = du \end{array}}$

$$= \frac{\ln|u-1|}{3} - \frac{1}{3}\left(\int \frac{w}{w^2+\frac{3}{4}}\, dw + \frac{3}{2}\int \frac{1}{w^2+\frac{3}{4}}\, dw\right)$$

$$= \frac{\ln|u-1|}{3} - \frac{1}{3}\left(\frac{1}{2}\ln\left|w^2+\frac{3}{4}\right| + \frac{3}{2}\left(\frac{2}{\sqrt{3}}\right)\arctan\left(\frac{w}{\left(\frac{\sqrt{3}}{2}\right)}\right)\right) + C$$

$$= \frac{\ln|\sin x - 1|}{3} - \frac{1}{6}\ln\left|\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2(u+\frac{1}{2})}{\sqrt{3}}\right) + C$$

$$\boxed{= \frac{\ln|\sin x - 1|}{3} - \frac{1}{6}\ln\left|\left(\sin x+\frac{1}{2}\right)^2 + \frac{3}{4}\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2(\sin x+\frac{1}{2})}{\sqrt{3}}\right) + C}$$

OPTIONAL BONUS #4 Compute the following integral:

4. $\displaystyle\int \frac{x^5+x^4+37x^3+18x^2+81x+81}{x^4+18x^2+81}\,dx = \int (x+1) + \frac{19x^3}{x^4+18x^2+81}\,dx$

Long division yields:

$$\begin{array}{r} x \quad + \quad 1 \\ x^4+18x^2+81\overline{)x^5+x^4+37x^3+18x^2+81x+81} \\ \underline{-(x^5+18x^3+81x)} \\ x^4+19x^3+18x^2+81 \\ \underline{-(x^4+18x^2+81)} \\ 19x^3 \end{array}$$

$= \displaystyle\int (x+1) + \frac{19x^3}{(x^2+9)^2}\,dx = \frac{x^2}{2}+x+19\int \frac{x^3}{(x^2+9)^2}\,dx = \frac{x^2}{2}+x+19\int \frac{x\cdot x^2}{(x^2+9)^2}\,dx$

We have at least 3 ways to finish this integral:

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Substitute $\boxed{\begin{array}{l} u = x^2+9 \Rightarrow x^2 = u-9 \\ du = 2x\ dx \\ \frac{1}{2}du = x\ dx \end{array}}$

$= \displaystyle\frac{x^2}{2}+x+\frac{19}{2}\int \frac{u-9}{u^2}\,du = \frac{x^2}{2}+x+\frac{19}{2}\int \frac{u}{u^2}-\frac{9}{u^2}\,du = \frac{x^2}{2}+x+\frac{19}{2}\int \frac{1}{u}-\frac{9}{u^2}\,du$

$= \displaystyle\frac{x^2}{2}+x+\frac{19}{2}\left(\ln|u|+\frac{9}{u}\right)+C = \boxed{\frac{x^2}{2}+x+\frac{19}{2}\left(\ln|x^2+9|+\frac{9}{x^2+9}\right)+C}$

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OR use a Partial Fractions Decomposition:

$$\frac{x^3}{(x^2+9)(x^2+9)} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2}$$

Clearing the denominator yields:

$x^3 = (Ax+B)(x^2+9) + (Cx+D)$
$x^3 = Ax^3+Bx^2+(9A+C)x+9B+D$
so that $A+B=1,\ B=0,\ 9A+C=0$ and $9B+D=0$
Solve for $A=1,\ B=0,\ C=-9$ and $D=0$

Finally, $\displaystyle\frac{x^3}{(x^2+9)(x^2+9)} = \frac{x}{x^2+9} + \frac{-9x}{(x^2+9)^2}$

Finish the original integral,

$\displaystyle\frac{x^2}{2}+x+19\int \frac{x^3}{(x^2+9)^2}\,dx = \frac{x^2}{2}+x+19\int \frac{x}{x^2+9}+\frac{-9x}{(x^2+9)^2}\,dx$

$= \displaystyle\frac{x^2}{2}+x+19\left(\frac{1}{2}\ln|x^2+9| + \int \frac{-9x}{(x^2+9)^2}\,dx\right)$

Substitute $\boxed{\begin{array}{l} v = x^2 + 9 \\ \\ dv = 2x\ dx \\ \\ \frac{1}{2}dv = x\ dx \end{array}}$

$$= \frac{x^2}{2} + x + 19\left(\frac{1}{2}\ln|x^2+9| - \frac{9}{2}\int \frac{1}{v^2}\ dv\right) = \frac{x^2}{2} + x + 19\left(\frac{1}{2}\ln|x^2+9| + \frac{9}{2}\frac{1}{v}\right)$$

$$\boxed{= \frac{x^2}{2} + x + \frac{19}{2}\ln|x^2+9| + \frac{(19)(9)}{2}\left(\frac{1}{x^2+9}\right) + C}$$

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OR do a trigonometric substitution to finish the integral

$$= \frac{x^2}{2} + x + 19\int \frac{x^3}{(x^2+9)^2}\ dx = \frac{x^2}{2} + x + 19\int \frac{27\tan^3\theta}{(9\tan^2\theta+9)^2} 3\sec^2\theta d\ \theta$$

(Trig.) Substitute $\boxed{\begin{array}{l} x = 3\tan\theta \\ \\ dx = 3\sec^2\theta\ d\theta \end{array}}$

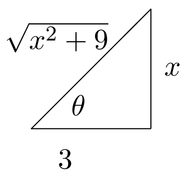
$$= \frac{x^2}{2} + x + 19\left(\frac{81}{81}\right)\int \frac{\tan^3\theta}{(\tan^2\theta+1)^2}\sec^2\theta\ d\theta = \frac{x^2}{2} + x + 19\int \frac{\tan^3\theta}{(\sec^2\theta)^2}\sec^2\theta\ d\theta$$

$$= \frac{x^2}{2} + x + 19\int \frac{\tan^3\theta}{\sec^4\theta}\sec^2\theta\ d\theta = \frac{x^2}{2} + x + 19\int \frac{\tan^3\theta}{\sec^2\theta}\ d\theta = \frac{x^2}{2} + x + 19\int \tan^3\theta\cos^2\theta\ d\theta$$

$$= \frac{x^2}{2} + x + 19\int \frac{\sin^3\theta}{\cos^3\theta}\cos^2\theta\ d\theta = \frac{x^2}{2} + x + 19\int \frac{\sin^3\theta}{\cos\theta}\ d\theta = \frac{x^2}{2} + x + 19\int \frac{\sin^2\theta}{\cos\theta}\sin\theta\ d\theta$$

$$= \frac{x^2}{2} + x + 19\int \frac{1-\cos^2\theta}{\cos\theta}\sin\theta\ d\theta = \frac{x^2}{2} + x - 19\int \frac{1-w^2}{w}\ dw = \frac{x^2}{2} + x - 19\int \frac{1}{w} - w\ dw$$

Substitute $\boxed{\begin{array}{l} w = \cos\theta \\ \\ dw = -\sin\theta\ d\theta \end{array}}$



$$= \frac{x^2}{2} + x - 19\left(\ln|w| - \frac{w^2}{2}\right) + C = \frac{x^2}{2} + x - 19\left(\ln|\cos\theta| - \frac{\cos^2\theta}{2}\right) + C$$

$$= \frac{x^2}{2} + x - 19\left(\ln\left|\frac{3}{\sqrt{x^2+9}}\right| - \frac{\left(\frac{3}{\sqrt{x^2+9}}\right)^2}{2}\right) + C$$

$$\boxed{= \frac{x^2}{2} + x - 19\left(\ln\left|\frac{3}{\sqrt{x^2+9}}\right| - \frac{9}{2(x^2+9)}\right) + C}$$