Math 12

Midterm Exam #2

March 31, 2010

Answer Key

1. [40 Points] Compute each of the following integrals, or else show that it diverges.

(a) $\displaystyle\int_0^9 \frac{1}{\sqrt{9-x}}\,dx = \lim_{t\to 9^-}\int_0^t \frac{1}{\sqrt{9-x}}\,dx = \lim_{t\to 9^-} -\int_9^{9-t}\frac{1}{\sqrt{u}}\,du = \lim_{t\to 9^-} -2\sqrt{u}\Big|_9^{9-t}$

$\displaystyle = \lim_{t\to 9^-} -2\sqrt{9-t} - (-2\sqrt{9}) = 0 - (-2\cdot 3) = \boxed{6}$

Substitute

$u = 9 - x$
$du = -xdx$
$-du = xdx$

$x = 0 \Rightarrow u = 9$
$x = t \Rightarrow u = 9 - t$

(b) $\displaystyle\int \frac{5}{(x-2)(x+3)}\,dx = \int \frac{1}{x-2} - \frac{1}{x+3}\,dx = \boxed{\ln|x-2| - \ln|x+3| + C}$

Partial Fractions Decomposition:

$\displaystyle\frac{5}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$

Clearing the denominator yields:

$5 = A(x+3) + B(x-2)$
$5 = (A+B)x + (3A - 2B)$
so that $A + B = 0$ and $3A - 2B = 5$
Solve for $A = 1$ and $B = -1$

(c) $\displaystyle\int \frac{x^3-1}{x^2+1}\,dx = \int x - \frac{x+1}{x^2+1}\,dx = \int x - \frac{x}{x^2+1} - \frac{1}{x^2+1}\,dx = \boxed{\frac{x^2}{2} - \frac{\ln|x^2+1|}{2} - \arctan x + C}$

Long division yields:

$$\begin{array}{r} x \\ x^2+1 \overline{)\, x^3 - 1} \\ \underline{-(x^3+x)} \\ -x-1 \end{array}$$

(d) $\displaystyle\int_{-\infty}^{\infty} e^x\,dx = \int_{-\infty}^{0} e^x\,dx + \int_{0}^{\infty} e^x\,dx = \lim_{s\to-\infty}\int_{s}^{0} e^x\,dx + \lim_{t\to\infty}\int_{0}^{t} e^x\,dx$

$$\lim_{s\to-\infty} e^x\Big|_s^0 + \lim_{t\to\infty} e^x\Big|_0^t = \lim_{s\to-\infty} e^0 - e^s + \lim_{t\to\infty} e^t - e^0 = (1-0) + (\infty - 1) = \infty$$

The second integral Diverges, so the entire integral $\boxed{\text{Diverges}}$.

(e) $\displaystyle\int_{7}^{\infty} \frac{1}{x^2-6x+25}\,dx = \lim_{t\to\infty}\int_{7}^{t} \frac{1}{x^2-6x+25}\,dx = \lim_{t\to\infty}\int_{7}^{t} \frac{1}{(x-3)^2+16}\,dx$ complete the square

Substitute $\boxed{\begin{array}{c} u = x-3 \\ du = dx \end{array}}$ $\boxed{\begin{array}{c} x = 7 \Rightarrow u = 4 \\ x = t \Rightarrow u = t-3 \end{array}}$

$$= \lim_{t\to\infty}\int_{4}^{t-3} \frac{1}{u^2+16}\,du = \lim_{t\to\infty}\frac{1}{16}\int_{4}^{t-3} \frac{1}{\left(\frac{u}{4}\right)^2+1}\,du = \lim_{t\to\infty}\frac{4}{16}\int_{1}^{\frac{t-3}{4}} \frac{1}{v^2+1}\,dv$$

Substitute $\boxed{\begin{array}{c} v = \frac{u}{4} \\ dv = \frac{1}{4}du \\ 4dv = du \end{array}}$ $\boxed{\begin{array}{c} u = 4 \Rightarrow v = 1 \\ u = t-3 \Rightarrow v = \frac{t-3}{4} \end{array}}$

$$= \lim_{t\to\infty}\frac{1}{4}\arctan v\Big|_1^{\frac{t-3}{4}} = \lim_{t\to\infty}\frac{1}{4}\left(\arctan\left(\frac{t-3}{4}\right) - \arctan 1\right) = \frac{1}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{1}{4}\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{16}}$$

2. [6 Points] Determine whether the following sequence **converges** or **diverges**. If it converges, compute its limit. Justify your answer. Do not just put down a number.

$$\left\{n^{\frac{1}{n}}\right\}_{n=1}^{\infty}$$

We switch to the variable x and the related function $f(x) = x^{\frac{1}{x}}$ in order to apply L'H Rule:

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln\left(x^{\frac{1}{x}}\right)} = e^{\lim\limits_{x\to\infty}\ln\left(x^{\frac{1}{x}}\right)} = e^{\lim\limits_{x\to\infty}\frac{\ln x}{x}} = e^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{1}} = e^0 = \boxed{1}$$

Finally, $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$, so that the sequence $\left\{n^{\frac{1}{n}}\right\}_{n=1}^{\infty}$ converges to 1.

3. [8 Points] Find the **sum** of the following series (which does converge):

$$\sum_{n=1}^{\infty}\frac{(-1)^n 2^{n-1}}{3^{2n+1}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n-1}}{3^{2n+1}} = -\frac{1}{3^3} + \frac{2}{3^5} - \frac{2^2}{3^7} + \dots$$

Here we have a nice geometric series with $a = -\dfrac{1}{27}$ and $r = -\dfrac{2}{9}$

As a result, the sum is given by $\dfrac{a}{1-r} = \dfrac{-\frac{1}{27}}{1-\left(-\frac{2}{9}\right)} = \dfrac{-\frac{1}{27}}{\frac{11}{9}} = -\dfrac{1}{27} \cdot \dfrac{9}{11} = \boxed{-\dfrac{1}{33}}$

4. [18 Points] Determine whether each of the following series **converges** or **diverges**. Name any convergence test(s) you use, and justify that it's legal to use them. Show all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^7 + 6n^{\frac{3}{2}} + 5}{8n^9 - \sqrt{n} + 441} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series with $p = 2 > 1$.

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{3n^7 + 6n^{\frac{3}{2}} + 5}{8n^9 - \sqrt{n} + 441}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^9 + 6n^{\frac{7}{2}} + 5n^2}{8n^9 - \sqrt{n} + 441} = \frac{3}{8}$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then the original series also $\boxed{\text{Converges by Limit Comparison Test}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2+1}$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$

since $\displaystyle\lim_{n\to\infty} \frac{e^n}{n^2+1} = \infty$ because $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2+1} \overset{\frac{\infty}{\infty}\ \text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\frac{\infty}{\infty}\ \text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$.

(c) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{7}{8}\right)^n$ is a $\boxed{\text{Convergent Geometric Series}}$ with $|r| = \left|-\dfrac{7}{8}\right| = \dfrac{7}{8} < 1$.

5. [28 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+5}$

First, we show the absolute series is divergent. Note that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+5} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent p-series with $p = 1$. Next, Check: $\lim_{n\to\infty} \dfrac{\dfrac{1}{n+5}}{\dfrac{1}{n}} = \dfrac{n}{n+5} = 1$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent, then $\sum_{n=1}^{\infty} \dfrac{1}{n+5}$ is also divergent by Limit Comparison Test. As a result, we have no chance for Absolute Convergence.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

- $b_n = \dfrac{1}{n+5} > 0$
- $\lim_{n\to\infty} \dfrac{1}{n+5} = 0$
- $\dfrac{1}{b_{n+1}} < \dfrac{1}{b_n}$ because $\dfrac{1}{n+6} < \dfrac{1}{n+5}$

Therefore, the original series converges by the Alternating Series Test. Finally, we can conclude the original series is $\boxed{\text{Conditionally Convergent}}$

(b) $\sum_{n=1}^{\infty} \dfrac{(-2)^n n!}{n^n}$

Try Ratio Test:

$$\lim_{n\to\infty} \left| \dfrac{\dfrac{(-2)^{n+1}(n+1)!}{(n+1)^{n+1}}}{\dfrac{(-2)^n n!}{n^n}} \right| = \lim_{n\to\infty} \left| \dfrac{(-2)^{n+1}}{(-2)^n} \right| \cdot \dfrac{(n+1)!}{n!} \cdot \dfrac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} 2 \cdot \dfrac{(n+1)n!}{n!} \cdot \dfrac{n^n}{(n+1)(n+1)^n} = \lim_{n\to\infty} 2 \cdot \dfrac{n^n}{(n+1)^n} = \lim_{n\to\infty} 2 \cdot \left(\dfrac{n}{n+1}\right)^n = \dfrac{2}{e} < 1$$

The series is $\boxed{\text{Absolutely Convergent by the Ratio Test}}$.

(c) $\sum_{n=1}^{\infty} (-1)^n \dfrac{\arctan n}{4^n}$

First examine the absolute series $\sum_{n=1}^{\infty} \dfrac{\arctan n}{4^n}$.

Note that $\sum_{n=1}^{\infty} \dfrac{1}{4^n}$ is a convergent Geometric Series with $|r| = \dfrac{1}{4} < 1$, so $\dfrac{\pi}{2} \sum_{n=1}^{\infty} \dfrac{1}{4^n}$ is also convergent.

Since the terms are bounded $\dfrac{\arctan n}{4^n} \leq \dfrac{\pi}{2} \cdot \dfrac{1}{4^n}$ and $\sum_{n=1}^{\infty} \dfrac{\pi}{2} \dfrac{1}{4^n}$ is convergent, then the series $\sum_{n=1}^{\infty} \dfrac{\arctan n}{4^n}$ is also $\boxed{\text{Convergent by Comparison Test}}$.

Finally, the original series $\sum_{n=1}^{\infty}(-1)^n\frac{\arctan n}{4^n}$ is $\boxed{\text{Absolutely Convergent by Absolute Convergence Test}}$.

(d) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ $\boxed{\text{Converges Absolutely by Absolute Convergence Test}}$ since the absolute series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent $p-$series with $p = 3 > 1$.

OPTIONAL BONUS #1

Compute the sum of the following series:

1. $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2+2n}\right) = \sum_{n=1}^{\infty}\ln\left(\frac{n^2+2n}{n^2+2n}+\frac{1}{n^2+2n}\right) = \sum_{n=1}^{\infty}\ln\left(\frac{n^2+2n+1}{n^2+2n}\right) = \sum_{n=1}^{\infty}\ln\left(\frac{(n+1)^2}{n(n+2)}\right)$

$= \sum_{n=1}^{\infty}\ln\left(\frac{(n+1)(n+1)}{n(n+2)}\right) = \sum_{n=1}^{\infty}\ln\left[\left(\frac{n+1}{n}\right)\cdot\left(\frac{n+1}{n+2}\right)\right] = \sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)+\ln\left(\frac{n+1}{n+2}\right)$

$= \sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)-\ln\left(\frac{n+2}{n+1}\right)$

This series now looks telescoping, so we can examine the n^{th} partial sum

$S_n = \ln 2 - \ln\left(\frac{3}{2}\right)+\ln\left(\frac{3}{2}\right)-\ln\left(\frac{4}{3}\right)+\ln\left(\frac{4}{3}\right)-\ln\left(\frac{5}{4}\right)+\ln\left(\frac{5}{4}\right)-\ln\left(\frac{6}{5}\right)+\ldots-\ln\left(\frac{n+1}{n}\right)+$ $\ln\left(\frac{n+1}{n}\right)-\ln\left(\frac{n+2}{n+1}\right) = \ln 2-\ln\left(\frac{n+2}{n+1}\right)$

Note $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\ln 2-\ln\left(\frac{n+2}{n+1}\right) = \ln 2 - \ln 1 = \ln 2$

Finally the sum of the original series is equal to the limit of the n^{th} partial sums. That is, $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2+2n}\right) = \lim_{n\to\infty} S_n = \boxed{\ln 2}$

OPTIONAL BONUS #2

Determine whether the following series converges or diverges.

2. $\sum_{n=1}^{\infty}\frac{(-1)^n n^{3n}}{n^3(n!)^2 e^{n^2}}$ Try Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(n+1)^{3n+3}}{(n+1)^3[(n+1)!]^2e^{(n+1)^2}}}{\dfrac{(-1)^n n^{3n}}{n^3[n!]^2e^{n^2}}}\right| = \lim_{n\to\infty}\frac{(n+1)^{3n+3}}{n^{3n}}\cdot\frac{n^3}{(n+1)^3}\cdot\frac{(n!)^2}{[(n+1)!]^2}\cdot\frac{e^{n^2}}{e^{n^2+2n+1}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^3(n+1)^{3n}}{n^{3n}}\cdot\left(\frac{n}{n+1}\right)^3\cdot\frac{(n!)^2}{(n+1)^2(n!)^2}\cdot\frac{e^{n^2}}{e^{n^2+2n+1}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^3(n+1)^{3n}}{n^{3n}}\cdot\left(\frac{n}{n+1}\right)^3\cdot\frac{1}{(n+1)^2}\cdot\frac{e^{n^2}}{e^{n^2}e^{2n+1}}$$

$$= \lim_{n\to\infty}(n+1)^3\left(\left(\frac{n+1}{n}\right)^n\right)^3\cdot(1)\cdot\frac{1}{(n+1)^2}\cdot\frac{1}{e^{2n+1}} = \lim_{n\to\infty}(n+1)e^3\cdot(1)\cdot\frac{1}{e^{2n+1}}$$

$$= \lim_{n\to\infty} \frac{n+1}{e^{2n-2}} = \frac{\infty}{\infty} \overset{=}{\text{L'H}} \lim_{n\to\infty} \frac{1}{2e^{2n-2}} = 0 < 1. \text{ So the series } \boxed{\text{Converges by the Ratio Test}}$$

OPTIONAL BONUS #3 Compute the following integral:

3. $\displaystyle\int \frac{\arctan x}{x^6}\, dx = \int \arctan x (x^{-6})\, dx = \arctan x \cdot \left(-\frac{1}{5x^5}\right) - \int \left(-\frac{1}{5x^5}\right) \frac{1}{x^2+1} dx$

I.B.P $\boxed{\begin{array}{ll} u = \arctan x & dv = x^{-6} dx \\ du = \frac{1}{x^2+1} dx & v = -\frac{1}{5x^5} \end{array}}$

$$= \left(-\frac{\arctan x}{5x^5}\right) + \frac{1}{5}\int \frac{1}{x^5(x^2+1)} dx = \left(-\frac{\arctan x}{5x^5}\right) + \frac{1}{5}\int \frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5} - \frac{x}{x^2+1} dx$$

Partial Fractions Decomposition:

$$\frac{1}{x^5(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x^5} + \frac{Fx+G}{x^2+1}$$

Clearing the denominator yields:

$1 = Ax^4(x^2+1) + Bx^3(x^2+1) + Cx^2(x^2+1) + Dx(x^2+1) + E(x^2+1) + (Fx+G)x^5$
$1 = (A+F)x^6 + (B+G)x^5 + (A+C)x^4 + (B+D)x^3 + (C+E)x^2 + Dx + E$
so that $A+F = 0, B+G = 0, A+C = 0, B+D = 0, C+E = 0, D = 0$ and $E = 1$.

Solve for $D = 0 \Rightarrow B = 0 \Rightarrow G = 0$ and $E = 1 \Rightarrow C = -1 \Rightarrow A = 1 \Rightarrow F = -1$

$$= \boxed{\left(-\frac{\arctan x}{5x^5}\right) + \frac{1}{5}\left(\ln|x| + \frac{1}{2x^2} - \frac{1}{4x^4} - \frac{1}{2}\ln|x^2+1|\right) + C}$$

OPTIONAL BONUS #4 Compute the following integral:

4. $\displaystyle\int \frac{16e^{3x}}{e^{4x}-16}\, dx = \int \frac{16\,(e^x)^2\, e^x}{(e^x)^4 - 16}\, dx = \int \frac{16u^2}{u^4-16}\, du = \int \frac{16u^2}{(u^2-4)(u^2+4)}\, du$

Substitute $\boxed{\begin{array}{l} u = e^x \\ du = e^x dx \end{array}}$

$$= \int \frac{16u^2}{(u-2)(u+2)(u^2+4)}\, du = \int \frac{2}{u-2} - \frac{2}{u+2} + \frac{8}{u^2+4}\, du$$

Partial Fractions Decomposition:

$$\frac{16u^2}{(u-2)(u+2)(u^2+4)} = \frac{A}{u-2} + \frac{B}{u+2} + \frac{Cu+D}{u^2+4}$$

Clearing the denominator yields:

$16u^2 = A(u+2)(u^2+4) + B(u-2)(u^2+4) + (Cu+D)(u^2+4)$
$16u^2 = A(u^3+2u^2+4u+8) + B(u^3-2u^2+4u-8) + (Cu^3+Du^2+4Cu+4D)$
so that $A+F=0, B+G=0, A+C=0, B+D=0, C+E=0, D=0$ and $E=1$.

Solve for $D=0 \Rightarrow B=0 \Rightarrow G=0$ and $E=1 \Rightarrow C=-1 \Rightarrow A=1 \Rightarrow F=-1$

$$= 2\ln|u-2| - 2\ln|u+2| + \frac{8}{2}\arctan\left(\frac{u}{2}\right) + C$$

$$\boxed{= 2\ln|e^x-2| - 2\ln|e^x+2| + 4\arctan\left(\frac{e^x}{2}\right) + C}$$