Math 12 Midterm Exam #1 February 23, 2011

Answer Key

1. [30 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to 0} \frac{xe^x - x}{x\sin(3x)} \overset{(\frac{0}{0})^{\text{L'H}}}{=} \lim_{x\to 0} \frac{xe^x + e^x - 1}{x3\cos(3x) + \sin(3x)}$$

$$\overset{(\frac{0}{0})^{\text{L'H}}}{=} \lim_{x\to 0} \frac{xe^x + e^x + e^x}{3x(-\sin(3x))3 + 3\cos(3x) + 3\cos(3x)} = \frac{0 + e^0 + e^0}{0+3+3} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

(b) $$\lim_{x\to\infty} \left(1 - \frac{3}{x}\right)^x \overset{1^\infty}{=} \lim_{x\to\infty} e^{\ln\left(\left(1-\frac{3}{x}\right)^x\right)} = e^{\lim\limits_{x\to\infty} x\ln\left(1-\frac{3}{x}\right)} \overset{\infty\cdot 0}{=} e^{\left(\lim\limits_{x\to\infty} \frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{x}}\right)}$$

$$\overset{(\frac{0}{0})^{\text{L'H}}}{=} e^{\left(\lim\limits_{x\to\infty} \frac{\frac{1}{\left(1-\frac{3}{x}\right)}\cdot\frac{3}{x^2}}{-\frac{1}{x^2}}\right)} = e^{\left(\lim\limits_{x\to\infty} \frac{-3x^2}{\left(1-\frac{3}{x}\right)x^2}\right)} = e^{\left(\lim\limits_{x\to\infty} \frac{-3}{\left(1-\frac{3}{x}\right)}\right)} = \boxed{e^{-3}}$$

(c) $$\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}} \overset{\infty^0}{=} \lim_{x\to\infty} e^{\ln\left((e^x+x)^{\frac{1}{x}}\right)} = e^{\lim\limits_{x\to\infty} \ln\left((e^x+x)^{\frac{1}{x}}\right)} = e^{\lim\limits_{x\to\infty} \frac{1}{x}\ln(e^x+x)}$$

$$= e^{\lim\limits_{x\to\infty} \frac{\ln(e^x+x)}{x}} \overset{(\frac{\infty}{\infty})^{\text{L'H}}}{=} e^{\left(\lim\limits_{x\to\infty} \frac{\left(\frac{e^x+1}{e^x+x}\right)}{1}\right)} = e^{\lim\limits_{x\to\infty} \frac{e^x+1}{e^x+x}} \overset{(\frac{\infty}{\infty})^{\text{L'H}}}{=} = e^{\lim\limits_{x\to\infty} \frac{e^x}{e^x+1}}$$

$$\overset{(\frac{\infty}{\infty})^{\text{L'H}}}{=} e^{\lim\limits_{x\to\infty} \frac{e^x}{e^x}} = e^{\lim\limits_{x\to\infty} 1} = e^1 = \boxed{e}$$

2. [30 Points] Compute each of the following **definite integrals**. Please simplify your answer.

(a) $$\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \int_1^{\sqrt{3}} \frac{1}{\sqrt{4(1-\frac{x^2}{4})}}\,dx = \int_1^{\sqrt{3}} \frac{1}{\sqrt{4(1-\left(\frac{x}{2}\right)^2)}}\,dx = \int_1^{\sqrt{3}} \frac{1}{\sqrt{4}\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

$$\int_1^{\sqrt{3}} \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = \frac{1}{2}\int_1^{\sqrt{3}} \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = \frac{2}{2}\int_{w=\frac{1}{2}}^{w=\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-w^2}}\,dw = \arcsin w\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{3} - \frac{\pi}{6} = \boxed{\frac{\pi}{6}}$$

Here $\boxed{\begin{aligned} w &= \tfrac{x}{2} \\ dw &= \tfrac{1}{2}dx \\ 2dw &= dx \end{aligned}}$ and $\boxed{\begin{aligned} x = 1 &\implies w = \tfrac{1}{2} \\ x = \sqrt{3} &\implies w = \tfrac{\sqrt{3}}{2} \end{aligned}}$

(b) $$\int_0^{\ln 7} \tanh x \; dx = \int_0^{\ln 7} \frac{\sinh x}{\cosh x} \; dx = \int_1^{\frac{25}{7}} \frac{1}{w} \; dw = \ln|w|\Big|_1^{\frac{25}{7}} = \ln\left(\frac{25}{7}\right) - \ln 1 = \boxed{\ln\left(\frac{25}{7}\right)}$$

Here $\boxed{\begin{aligned} w &= \cosh x \\ dw &= \sinh x dx \end{aligned}}$ and $\boxed{\begin{aligned} x = 0 &\implies w = 1 \\ x = \ln 7 &\implies w = \cosh(\ln 7) = \frac{e^{\ln 7} - \frac{1}{e^{\ln 7}}}{2} = \frac{7 - \frac{1}{7}}{2} = \frac{25}{7} \end{aligned}}$

OR $$\int_0^{\ln 7} \tanh x \; dx = \int_0^{\ln 7} \frac{\sinh x}{\cosh x} \; dx = \ln|\cosh x|\Big|_0^{\ln 7} = \ln(\cosh(\ln 7)) - \ln(\cosh 0)$$

$$= \ln\left(\frac{25}{7}\right) - \ln 1 = \boxed{\ln\left(\frac{25}{7}\right)}$$

(c) $$\int_1^e (\ln x)^2 \; dx$$

Method 1:

First I.B.P. $\boxed{\begin{aligned} u &= (\ln x)^2 & dv &= dx \\ du &= 2\ln x \frac{1}{x} dx & v &= x \end{aligned}}$ Second I.B.P. $\boxed{\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{1}{x} dx & v &= x \end{aligned}}$

$$\int_1^e (\ln x)^2 \; dx = x(\ln x)^2\Big|_1^e - 2\int_1^e \ln x \cdot \left(\frac{1}{x}\right) \cdot x \; dx = x(\ln x)^2\Big|_1^e - 2\int_1^e \ln x \; dx$$

$$= x(\ln x)^2\Big|_1^e - 2\left(x\ln x\Big|_1^e - \int_1^e \left(\frac{1}{x}\right) \cdot x \; dx\right) = x(\ln x)^2\Big|_1^e - 2\left(x\ln x\Big|_1^e - \int_1^e 1 \; dx\right)$$

$$= x(\ln x)^2\Big|_1^e - 2\left(x\ln x\Big|_1^e - x\Big|_1^e\right) = \left(e(\ln e)^2 - 1(\ln 1)^2\right) - 2\left((e\ln e - 1\ln 1) - (e-1)\right)$$

$$= (e - 0) - 2((e-0) - (e-1)) = e - 2(e - e + 1) = \boxed{e - 2}$$

Method 2:

First I.B.P. $\boxed{\begin{aligned} u &= \ln x & dv &= \ln x dx \\ du &= \frac{1}{x} dx & v &= x\ln x - x \text{ (by I.B.P)} \end{aligned}}$ $\ln x$ I.B.P. $\boxed{\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{1}{x} dx & v &= x \end{aligned}}$

$$\int_1^e (\ln x)^2 \; dx = (\ln x)(x\ln x - x)\Big|_1^e - \int_1^e (x\ln x - x) \cdot \left(\frac{1}{x}\right) \; dx = (\ln x)(x\ln x - x)\Big|_1^e - \int_1^e \ln x - 1 \; dx$$

$$= (\ln x)(x\ln x - x)\Big|_1^e - \left(\int_1^e \ln x\ dx - \int_1^e 1\ dx\right) = (\ln x)(x\ln x - x)\Big|_1^e - \left(\left(x\ln x\Big|_1^e - \int_1^e 1\ dx\right) - x\Big|_1^e\right)$$

$$= (\ln x)(x\ln x - x)\Big|_1^e - \left(\left(x\ln x\Big|_1^e - x\Big|_1^e\right) - x\Big|_1^e\right)$$

$$= [(\ln e)(e\ln e - e) - (\ln 1)(1\ln 1 - 1)] - [((e\ln e - 1\ln 1) - (e-1)) - (e-1)]$$

$$= [1(e-e) - 0(-1)] - [((e-0)-(e-1)) - (e-1)] = -[(e-e+1)-e+1] = -1+e-1 = \boxed{e-2}$$

3. [40 Points] Compute each of the following **indefinite integrals**.

(a) $\displaystyle\int \left(x + \frac{1}{e^{3x}}\right)^2\ dx = \int x^2 + \frac{2x}{e^{3x}} + \frac{1}{e^{6x}}\ dx = \frac{x^3}{3} + 2\int xe^{-3x}\ dx - \frac{1}{6}e^{-6x}\ dx$

$$= \frac{x^3}{3} + 2\left(-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\ dx\right) - \frac{1}{6}e^{-6x} = \frac{x^3}{3} + 2\left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right) - \frac{1}{6}e^{-6x} + C$$

$$= \boxed{\frac{x^3}{3} - \frac{2}{3}xe^{-3x} - \frac{2}{9}e^{-3x} - \frac{1}{6}e^{-6x} + C}$$

I.B.P. above (on middle term)

$u = x$	$dv = e^{-3x}dx$
$du = dx$	$v = -\frac{1}{3}e^{-3x}$

(b) $\displaystyle\int x\ \arctan x\ dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\ dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\ dx$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\ dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\ dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x) + C = \boxed{\frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C}$$

$u = \arctan x$	$dv = xdx$
$du = \frac{1}{1+x^2}dx$	$v = \frac{x^2}{2}$

OR if you don't like the "slip-in/slip out" technique, use a tangent trig. substitution instead to finish the second piece of the I.B.P. $\displaystyle\int \frac{x^2}{1+x^2}\ dx$

$$\int \frac{x^2}{1+x^2}\ dx = \int \frac{\tan^2\theta}{1+\tan^2\theta}\sec^2\theta\ d\theta = \int \frac{\tan^2\theta}{\sec^2\theta}\sec^2\theta\ d\theta = \int \tan^2\theta\ d\theta = \int \sec^2\theta - 1\ d\theta$$

$$= \tan\theta - \theta = x - \arctan x$$

Trig. Substitute

$x = \tan\theta$
$dx = \sec^2\theta d\theta$

(c) $\displaystyle\int \frac{1}{(x^2+4)^{\frac{5}{2}}}\ dx = \int \frac{1}{(4\tan^2\theta+4)^{\frac{5}{2}}} \cdot\ 2\sec^2\theta\ d\theta = \int \frac{1}{(4\sec^2\theta)^{\frac{5}{2}}} \cdot\ 2\sec^2\theta\ d\theta$

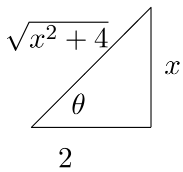
$$= \int \frac{1}{(\sqrt{4\sec^2\theta})^5} \cdot\ 2\sec^2\theta\ d\theta = \int \frac{1}{(2\sec\theta)^5} \cdot\ 2\sec^2\theta\ d\theta$$

$$= \frac{1}{2^4}\int \frac{\sec^2\theta}{\sec^5\theta}\ d\theta = \frac{1}{2^4}\int \frac{1}{\sec^3\theta}\ d\theta = \frac{1}{16}\int \cos^3\theta\ d\theta$$

$$= \frac{1}{16}\int \cos^2\theta\cos\theta\ d\theta\ = \frac{1}{16}\int (1-\sin^2\theta)\cos\theta\ d\theta = \frac{1}{16}\int (1-w^2)\ dw$$

$$= \frac{1}{16}\left(w - \frac{w^3}{3}\right) + C = \frac{1}{16}\left(\sin\theta - \frac{\sin^3\theta}{3}\right) + C = \frac{1}{16}\left(\frac{x}{\sqrt{x^2+4}} - \frac{1}{3}\left(\frac{x}{\sqrt{x^2+4}}\right)^3\right) + C$$

$$= \boxed{\frac{1}{16}\left(\frac{x}{\sqrt{x^2+4}} - \frac{x^3}{3(x^2+4)^{\frac{3}{2}}}\right) + C}$$



Trig. Substitute $\boxed{\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta d\theta \end{array}}$

Standard w substitution for *odd* trig. integral $\displaystyle\int \cos^3\theta\ d\theta$ technique:

$$\boxed{\begin{array}{l} w = \sin\theta \\ dw = \cos\theta\ d\theta \end{array}}$$

3. (Continued) Compute the following **indefinite integral**.

(d) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\ dx = \int \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}} 2\cos\theta\ d\theta = \int \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}} 2\cos\theta\ d\theta$

$$= 4\int \frac{\sin^2\theta}{\sqrt{4}\sqrt{\cos^2\theta}} 2\cos\theta\ d\theta = 4\int \frac{\sin^2\theta}{2\cos\theta} 2\cos\theta\ d\theta = 4\int \sin^2\theta\ d\theta$$

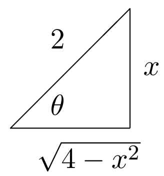
$$= 4\int \frac{1-\cos(2\theta)}{2}\ d\theta = 2\int 1-\cos(2\theta)\ d\theta = 2\left(\theta - \frac{\sin(2\theta)}{2}\right) + C$$

$$= 2\left(\theta - \frac{2\sin\theta\cos\theta}{2}\right) + C = 2\left(\theta - \sin\theta\cos\theta\right) + C$$

$$= 2\left(\arcsin\left(\frac{x}{2}\right) - \frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right) + C = \boxed{2\arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C}$$

Trig. Substitute

$$\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta d\theta \end{aligned}}$$



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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute the following **indefinite integral**.

1. $\int \frac{1}{\sqrt{e^x - 1}}\, dx$

OPTIONAL BONUS #2 Compute the following **indefinite integral**.

2. $\int \frac{xe^x}{\sqrt{1 + e^x}} dx$