Math 12

Midterm Exam #1

February 24, 2010

Answer Key

1. [6 Points] Compute the **derivative** for the following function. Do not simplify your answer.

$$f(x) = \arctan(4x) \cdot \arcsin(3x)$$

$$\boxed{f'(x) = \arctan(4x) \cdot \frac{1}{\sqrt{1-(3x)^2}} \cdot 3 + \arcsin(3x) \cdot \frac{1}{1+(4x)^2} \cdot 4}$$

2. [24 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim\limits_{x\to\infty} \dfrac{\arctan x}{\frac{1}{x}+1} = \dfrac{\frac{\pi}{2}}{1} = \boxed{\dfrac{\pi}{2}}$ by direct substitution property

(b) $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{9\cos x - 5x - 9} = \left(\dfrac{0}{0}\right) \overset{\text{L'H}}{=} \lim\limits_{x\to 0} \dfrac{3\cos(3x)}{-9\sin x - 5} = \boxed{-\dfrac{3}{5}}$

(c) $\lim\limits_{x\to\infty} x\left(2e^{\frac{1}{x}} - 2\right) = (\infty \cdot 0) = \lim\limits_{x\to\infty} \dfrac{2e^{\frac{1}{x}} - 2}{\frac{1}{x}} = \left(\dfrac{0}{0}\right) \overset{\text{L'H}}{=} \lim\limits_{x\to\infty} \dfrac{2e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} 2e^{\frac{1}{x}} = 2e^0 =$
$\boxed{2}$

(d) $\lim\limits_{x\to 0} (1+3x)^{\frac{2}{x}} = \lim\limits_{x\to 0} e^{\ln\left((1+3x)^{\frac{2}{x}}\right)} = e^{\lim\limits_{x\to 0} \ln\left((1+3x)^{\frac{2}{x}}\right)} = e^{\lim\limits_{x\to 0} \frac{2\ln(1+3x)}{x}} = \left(e^{\frac{0}{0}}\right)$

$\overset{\text{L'H}}{=} e^{\lim\limits_{x\to 0} \frac{\frac{2}{1+3x}\cdot 3}{1}} = \boxed{e^6}$

3. [25 Points] Compute each of the following **definite integrals**. Please simplify your answer.

(a) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x\Big|_0^{\frac{1}{2}} = \arcsin\left(\frac{1}{2}\right) - \arcsin 0 = \frac{\pi}{6} - 0 = \boxed{\frac{\pi}{6}}$

(b) $\displaystyle\int_1^e \ln x\,dx \overset{\text{I.B.P}}{=} x\ln x\Big|_1^e - \int_1^e x\cdot\frac{1}{x}\,dx = x\ln x\Big|_1^e - x\Big|_1^e = e\ln e - 1\ln 1 - (e-1) = e - e + 1 = \boxed{1}$

$u = \ln x$	$dv = dx$
$du = \frac{1}{x}dx$	$v = x$

(c) $\displaystyle\int_0^{\ln 3} \sinh x\,dx = \cosh x\Big|_0^{\ln 3} = \cosh(\ln 3) - \cosh 0 = \frac{1}{2}\left(e^{\ln 3} + \frac{1}{e^{\ln 3}}\right) - 1 = \frac{1}{2}\left(3 + \frac{1}{3}\right) - 1$

$\displaystyle = \frac{1}{2}\left(\frac{10}{3}\right) - 1 = \left(\frac{5}{3}\right) - 1 = \boxed{\frac{2}{3}}$

4. [45 Points] Compute each of the following **indefinite integrals**.

(a) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx = \int \frac{1}{1+u^2}\,du = \arctan u + C = \boxed{\arctan e^x + C}$

Substitute

$u = e^x$
$du = e^x dx$

(b) $\displaystyle\int x^2 e^x\,dx \overset{\text{I.B.P}}{=} x^2e^x - 2\int xe^x dx \overset{\text{I.B.P}}{=} x^2e^x - 2\left[xe^x - \int e^x\,dx\right] = \boxed{x^2e^x - 2xe^x + 2e^x + C}$

$u = x^2$	$dv = e^x dx$
$du = 2xdx$	$v = e^x$

$u = x$	$dv = e^x dx$
$du = dx$	$v = e^x$

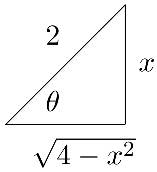
4. (Continued) Compute each of the following **indefinite integrals**.

(c) $\displaystyle\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\ dx = \int \frac{1}{(4-4\sin^2\theta)^{\frac{3}{2}}} \cdot\ 2\cos\theta\ d\theta = \int \frac{1}{(4\cos^2\theta)^{\frac{3}{2}}} \cdot\ 2\cos\theta\ d\theta$

$\displaystyle= \int \frac{1}{(2\cos\theta)^3} \cdot\ 2\cos\theta\ d\theta = \int \frac{1}{8\cos^3\theta} \cdot\ 2\cos\theta\ d\theta = \frac{1}{4}\int \frac{1}{\cos^2\theta}\ d\theta = \frac{1}{4}\int \sec^2\theta\ d\theta = \frac{1}{4}\tan\theta + C$

$\displaystyle= \boxed{\frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C}$

Trig. Substitute $\boxed{\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta d\theta \end{array}}$



(d) $\displaystyle\int \sin^3 x\ \cos^4 x\ dx = \int \sin^2 x\ \cos^4 x \sin x\ dx = \int (1-\cos^2 x)\ \cos^4 x \sin x\ dx$

$\displaystyle= -\int (1-u^2)u^4\ du = -\int u^4 - u^6\ du = \int -u^4 + u^6\ du = -\frac{u^5}{5} + \frac{u^7}{7} + C = \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C}$

Substitute $\boxed{\begin{array}{l} u = \cos x \\ dx = -\sin x dx \\ -dx = \sin x dx \end{array}}$

(e) $\displaystyle\int x\ \arcsin x\ dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\ dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\ d\theta$

$\displaystyle= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta\ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta\ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta d\theta$

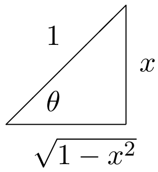
$\displaystyle= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2} d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1 - \cos(2\theta) d\theta$

$\displaystyle= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$

$\displaystyle= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}2\sin\theta\cos\theta + C = \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x dx \\ du = \frac{1}{\sqrt{1-x^2}} dx & v = \frac{x^2}{2} \end{array}}$$

Trig. Substitute $\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta d\theta \end{array}}$



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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute the following **indefinite integral**.

1. $\displaystyle\int \frac{x^3}{1-\sin(x^2)}\, dx = \int \frac{x^2\ x}{1-\sin(x^2)}\, dx = \frac{1}{2}\int \frac{w}{1-\sin w}\, dw = \frac{1}{2}\int \frac{w}{1-\sin w}\cdot\frac{1+\sin w}{1+\sin w}\, dw$

$\displaystyle = \frac{1}{2}\int \frac{w(1+\sin w)}{1-\sin^2 w}\, dw = \frac{1}{2}\int \frac{w(1+\sin w)}{\cos^2 w}\, dw = \frac{1}{2}\int \frac{w + w\sin w}{\cos^2 w}\, dw = \frac{1}{2}\int \frac{w}{\cos^2 w} + \frac{w\sin w}{\cos^2 w}\, dw$

$\displaystyle = \frac{1}{2}\int w\sec^2 w + w\cdot\frac{\sin w}{\cos w}\cdot\frac{1}{\cos w}\, dw = \frac{1}{2}\int w\sec^2 w + w\sec w\tan w\, dw$

$\displaystyle \overset{\text{two}}{\underset{\text{I.B.P}}{=}} \frac{1}{2}\left[\left(w\tan w - \int \tan w\, dw\right) + \left(w\sec w + \int \sec w\, dw\right)\right]$

$\displaystyle = \frac{1}{2}\left[(w\tan w + \ln|\cos w|) + (w\sec w + \ln|\sec w + \tan w|)\right] + C$

$\displaystyle = \boxed{\frac{1}{2}\left[x^2\tan x^2 + \ln|\cos x^2| + x^2\sec x^2 + \ln|\sec x^2 + \tan x^2|\right] + C}$

Substitute $\boxed{\begin{array}{l} w = x^2 \\ dw = 2x dx \\ \frac{1}{2} dw = x dx \end{array}}$ $\boxed{\begin{array}{ll} u = w & dv = \sec w dw \\ du = dw & v = \tan w \end{array}}$ $\boxed{\begin{array}{ll} u = w & dv = \sec w \tan w dw \\ du = dw & v = secw \end{array}}$

OPTIONAL BONUS #2 Compute the following **indefinite integral**.

2. $\displaystyle\int \frac{1}{x^{\frac{3}{2}}\left(1+x^{\frac{1}{3}}\right)}dx = \int \frac{1}{x^{\frac{3}{2}}\left(1+\left(x^{\frac{1}{6}}\right)^2\right)}dx = 6\int \frac{1}{u^9\left(1+u^2\right)}u^5\ du = 6\int \frac{1}{u^4\left(1+u^2\right)}\ du$

$\displaystyle= 6\int \frac{1}{\tan^4\left(1+\tan^2\theta\right)}\sec^2\theta\ d\theta$

$\displaystyle= 6\int \frac{1}{\tan^4\left(\sec^2\theta\right)}\sec^2\theta\ d\theta = 6\int \frac{1}{\tan^4\theta}\ d\theta = 6\int \cot^4\theta\ d\theta = 6\int \cot^2\theta\cot^2\theta\ d\theta$

$\displaystyle= 6\int \cot^2\theta(\csc^2\theta - 1)\ d\theta = 6\int \cot^2\theta\csc^2\theta - \cot^2\theta\ d\theta$

$\displaystyle= 6\int cot^2\theta\csc^\theta - (\csc^2\theta - 1)\ d\theta = 6\left(-\frac{\cot^3\theta}{3} + \cot\theta + \theta\right) + C = 6\left(-\frac{1}{3u^3} + \frac{1}{u} + \arctan u\right) + C$

$\displaystyle= 6\left(-\frac{1}{3\left(x^{\frac{1}{6}}\right)^3} + \frac{1}{x^{\frac{1}{6}}} + \arctan x^{\frac{1}{6}}\right) + C$

$\displaystyle= \boxed{6\left(-\frac{1}{3x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{6}}} + \arctan\left(x^{\frac{1}{6}}\right)\right) + C}$

Substitute

$u = x^{\frac{1}{6}}$

$du = \frac{1}{6}x^{-\frac{5}{6}}dx$

$6x^{\frac{5}{6}}du = dx$

$6\left(x^{\frac{1}{6}}\right)^5 du = dx$

$6u^5du = dx$

As a result $x^{\frac{3}{2}} = \left(x^{\frac{1}{6}}\right)^9 = u^9$ and $x^{\frac{1}{3}} = \left(x^{\frac{1}{6}}\right)^2 = u^2$.

Trig. Substitute

$u = tan\theta$

$du = \sec^2\theta d\theta$