Answer Key | Math 12 (Practice) Final Exam (May 14, 2009)

1. a. $\lim_{x\to 3} \frac{\ln(x-2)}{x^2-3x} \overset{0/0}{\overset{L'H}{=}} \lim_{x\to 3} \frac{\frac{1}{x-2}}{2x-3} = \boxed{\frac{1}{3}}$

b. $\lim_{x\to 0} \frac{\sinh x}{x} \overset{0/0}{\overset{L'H}{=}} \lim_{x\to 0} \frac{\cosh x}{1} = \boxed{1}$

c. $\lim_{x\to 0} x\cot x = \lim_{x\to 0} \frac{x\cos x}{\sin x} \frac{0}{0} = \lim_{x\to 0} \frac{x(-\sin x)+\cos x}{\cos x} = \frac{1}{1} = \boxed{1}$

2. a. $\int \frac{x+3}{\sqrt{9-x^2}}dx = \int \frac{x}{\sqrt{9-x^2}}dx + \int \frac{3}{\sqrt{9-x^2}} = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du + 3\int \frac{1}{\sqrt{9(1-(x/3)^2)}}dx$

$u^{-1/2}$

$\boxed{\begin{aligned} u &= 9-x^2 \\ du &= -2x\,dx \\ -\tfrac{1}{2}du &= x\,dx \end{aligned}}$

$= -\frac{1}{\cancel{2}}\cdot\cancel{2}u^{1/2} + \frac{\cancel{3}}{\cancel{3}}\int \frac{1}{\sqrt{1-(x/3)^2}}dx$

$\boxed{\begin{aligned} w &= x/3 \\ dw &= \tfrac{1}{3}dx \\ 3dw &= dx \end{aligned}}$

$= -\sqrt{9-x^2} + 3\int \frac{1}{\sqrt{1-w^2}}dw = \sqrt{9-x^2} + 3\arcsin w + C$

$= \boxed{\sqrt{9-x^2} + 3\arcsin\left(\frac{x}{3}\right) + C}$

b. $\int \frac{dx}{x^3+x^2-2x} = \int \frac{1}{x(x^2+x-2)}dx = \int \frac{1}{x(x+2)(x-1)}dx = \int -\frac{1}{2}\frac{1}{x} + \frac{1}{6}\frac{1}{x+2} + \frac{1}{3}\frac{1}{x-1}\,dx$

$= \boxed{-\frac{1}{2}\ln|x| + \frac{1}{6}\ln|x+2| + \frac{1}{3}\ln|x-1| + C}$

Partial Fractions Decomposition

$\left(\frac{1}{\cancel{x(x+2)(x-1)}} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}\right) x(x+2)(x-1)$

$\Rightarrow \quad 1 = A(x+2)(x-1) + B(x)(x-1) + C(x)(x+2)$

$= A(x^2+x-2) + B(x^2-x) + C(x^2+2x)$

$= Ax^2+Ax-2A + Bx^2-Bx + Cx^2+2Cx$

$1 = (A+B+C)x^2 + (A-B+2C)x - 2A$

$\Rightarrow$ • $A+B+C=0$ $\quad -\frac{1}{2}+B+C \quad B = +\frac{1}{2}-C$

• $A-B+2C=0$ $\quad -\frac{1}{2}-B+2C=0$

• $-2A=1 \Rightarrow \boxed{A=-\frac{1}{2}}$ $\quad -\frac{1}{2}-(+\frac{1}{2}-C)+2C=0$

$-1+C+2C=0 \quad \Rightarrow B = \frac{1}{2}-\frac{1}{3} = \boxed{\frac{1}{6}=B}$

$3C=1 \Rightarrow \boxed{C=\frac{1}{3}}$

2c. $\displaystyle\int x\sec^2 x\,dx \overset{\text{I.B.P.}}{=} x\tan x - \int \tan x\,dx = x\tan x - \int \frac{\sin x}{\cos x}\,dx = x\tan x + \int \frac{1}{u}\,du$

$$\boxed{\begin{array}{ll} u = x & dv = \sec^2 x\,dx \\ du = dx & v = \tan x \end{array}} \qquad \boxed{\begin{array}{l} u = \cos x \\ du = -\sin x \end{array}}$$

$= x\tan x + \ln|u| + C$

$= \boxed{x\tan x + \ln|\cos x| + C}$

3a. $\displaystyle\int_1^\infty \frac{dx}{x^2-2x+5} = \lim_{t\to\infty}\int_1^t \frac{1}{(x-1)^2+4}\,dx = \lim_{t\to\infty}\int_0^{t-1}\frac{1}{u^2+4}\,du = \lim_{t\to\infty}\frac{1}{4}\int_0^{t-1}\frac{1}{\left(\frac{u}{2}\right)^2+1}\,du$

$$\boxed{\begin{array}{l} u = x-1 \\ du = dx \end{array}} \quad \boxed{\begin{array}{l} x=1 \Rightarrow u=0 \\ x=t \Rightarrow u=t-1 \end{array}} \qquad \boxed{\begin{array}{l} w = u/2 \\ dw = \frac{1}{2}du \\ 2dw = du \end{array}} \quad \boxed{\begin{array}{l} u=0 \Rightarrow w= \\ u=t-1 \Rightarrow w=t \end{array}}$$

Complete the Square:
$x^2-2x+5 = \underbrace{(x-1)^2}_{x^2-2x+1} + 4$

$\displaystyle = \lim_{t\to\infty}\frac{2}{4}\int_0^{\frac{t-1}{2}}\frac{1}{w^2+1}\,dw = \lim_{t\to\infty}\frac{1}{2}\arctan w\Big|_0^{\frac{t-1}{2}}$

$\displaystyle = \lim_{t\to\infty}\frac{1}{2}\left[\underbrace{\arctan\left(\frac{t-1}{2}\right)}_{\to \pi/2} - \underbrace{\arctan 0}_{\to 0}\right] = \frac{1}{2}\left[\frac{\pi}{2}\right] = \boxed{\frac{\pi}{4}}$ converges

3b. $\displaystyle\int_0^9 \frac{dx}{(x-1)^{4/3}} = \int_0^1 \frac{1}{(x-1)^{4/3}}\,dx + \int_1^9 \frac{1}{(x-1)^{4/3}}\,dx = \lim_{s\to 1^-}\int_0^s \frac{1}{(x-1)^{4/3}}\,dx + \lim_{t\to 1^+}\int_t^9 \frac{1}{(x-1)^{4/3}}$

undefined at $x=1$

Do quickly or u-sub.
$u = x-1$
$du = dx$

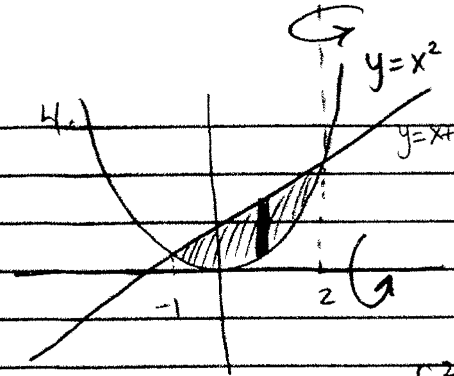
$\displaystyle = \lim_{s\to 1^-}\int_{-1}^{s-1} u^{-4/3}\,du + \lim_{t\to 1^+}\int_{t-1}^{8} u^{-4/3}\,du$

$\displaystyle = \lim_{s\to 1^-} -3u^{-1/3}\Big|_{-1}^{s-1} + \lim_{t\to 1^+} -3u^{-1/3}\Big|_{t-1}^{8}$

$\displaystyle = \lim_{s\to 1^-}\left[\underbrace{\frac{-3}{(s-1)^{1/3}}}_{\to\infty} - \left(\frac{-3}{(-1)^{1/3}}\right)\right] + \lim_{t\to 1^+}\left[\frac{-3}{8^{1/3}} - \underbrace{\left(\frac{-3}{(t-1)^{1/3}}\right)}_{\to -\infty}\right]$

$\Rightarrow$ $\boxed{\text{Diverges}}$

4.



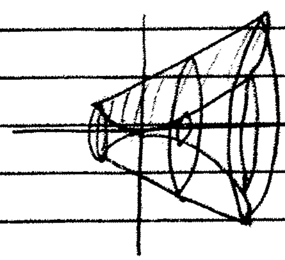
Interect $x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

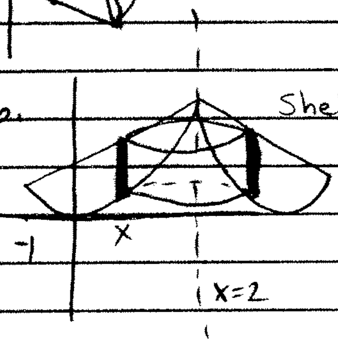
$\Rightarrow x = 2 \quad x = -1$

a. Washers $V = \displaystyle\int_{-1}^{2} \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] dx$



$$= \boxed{\int_{-1}^{2} \pi \left[(x+2)^2 - (x^2)^2\right] dx}$$

b.



Shells $V = \displaystyle\int_{-1}^{2} 2\pi \text{ radius} \cdot \text{height } dx$

$$= 2\pi \int_{-1}^{2} (2-x) \cdot \left[(x+2) - x^2\right] dx$$

5. $x = \sin^3 t \quad y = \cos^3 t \quad 0 \le t \le \pi/2$ (this should look familiar from a Review Packet)

$\frac{dx}{dt} = 3\sin^2 t \cos t \quad \frac{dy}{dt} = 3\cos^2 t(-\sin t)$

a $\displaystyle \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\cancel{3}\cos^{\cancel{2}} t \sin t}{\cancel{3}\sin^{\cancel{2}} t \cancel{\cos t}} = \frac{-\cos t}{\sin t}$ $\qquad \displaystyle \left.\frac{dy}{dx}\right|_{t=\pi/3} = \frac{-\cos \pi/3}{\sin \pi/3} = -\frac{1/2}{\sqrt{3}/2} = \boxed{\frac{-1}{\sqrt{3}}}$ slope

Point $\left(\frac{3\sqrt{3}}{8}, \frac{1}{8}\right) \Rightarrow y = \cos^3 t = \frac{1}{8}$

$\Rightarrow \cos t = \frac{1}{2}$

$\Rightarrow t = \pi/3$

Point Slope Form

$y - 1/8 = \frac{-1}{\sqrt{3}}\left(x - \frac{3\sqrt{3}}{8}\right)$

$y = -\frac{1}{\sqrt{3}} x + \frac{3}{8} + \frac{1}{8}$

$\Rightarrow \boxed{y = -\frac{1}{\sqrt{3}} x + 1/2}$

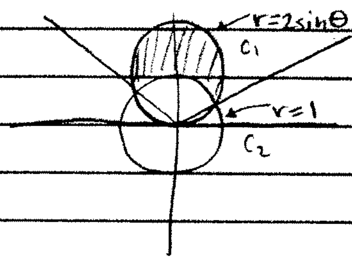
5b. Arclength $L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi/2} \sqrt{(3\sin^2 t\cos t)^2 + (-3\cos^2 t\sin t)^2}\, dt$

$$= \int_0^{\pi/2} \sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t}\, dt = \int_0^{\pi/2} \sqrt{9\sin^2 t\cos^2 t\overbrace{(\sin^2 t + \cos^2 t)}^{1}}\, dt$$

$$= \int_0^{\pi/2} 3\underbrace{\sin t\cos t}_{\text{already } \oplus \text{ on this interval}}\, dt = 3\left.\frac{\sin^2 t}{2}\right|_0^{\pi/2} = 3\left[\frac{\overbrace{\sin^2(\pi/2)}^{1}}{2} - \frac{\overbrace{\sin^2 0}^{0}}{2}\right] = \boxed{\frac{3}{2}}$$

can do u-sub if needed or note $\sin t\cos t = \frac{\sin(2t)}{2}$ then u-sub.

6 $r = 2\sin\theta$ $0 \le \theta \le \pi$ one loop of circle radius 1 centered at $(0,1)$



$r = 1$ circle radius 1 centered a origin (pole)

Intersect $2\sin\theta = 1$

$\Rightarrow \sin\theta = 1/2$

$\Rightarrow \theta = \pi/6$ and $5\pi/6$

Use Symmetry, integrate from $\theta = \pi/6$ to $\theta = \pi/2$ and Double

$$A = \cancel{2}\left[\int_{\pi/6}^{\pi/2} \frac{1}{\cancel{2}}\left[(\text{outer } r)^2 - (\text{inner } r)^2\right] d\theta\right] = \int_{\pi/6}^{\pi/2} (2\sin\theta)^2 - (1)^2\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} 4\sin^2\theta - 1\, d\theta = \int_{\pi/6}^{\pi/2} 4\left[\frac{1-\cos(2\theta)}{2}\right] - 1\, d\theta = \int_{\pi/6}^{\pi/2} 2 - 2\cos(2\theta) - 1\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} 1 - 2\cos(2\theta)\, d\theta = \theta - \sin(2\theta)\Big|_{\pi/6}^{\pi/2} = \left(\pi/2 - \overbrace{\sin\pi}^{0}\right) - \left(\pi/6 - \overbrace{\sin(\pi/3)}^{\sqrt{3}/2}\right)$$

do u-sub. if needed

$$= \pi/2 - \pi/6 + \sqrt{3}/2$$

$$= \frac{3\pi}{6} - \pi/6 + \sqrt{3}/2$$

$$= 2\pi/6 + \sqrt{3}/2$$

$$= \boxed{\pi/3 + \frac{\sqrt{3}}{2}}$$

7. Skip #7. We didn't cover this type!!

8\. a. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n+10)}{n^2+10n} \longrightarrow$ look at absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos(n+10)|}{n^2+10n}$

$$|\cos(n+10)| \le 1 \Rightarrow \frac{|\cos(n+10)|}{n^2+10n} \le \frac{1}{n^2+10n} \le \frac{1}{n^2}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent p-series $p=2>1$

Therefore, the smaller (absolute) series $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos(n+10)|}{n^2+10n}$ is convergent by C.T.,

and we have Absolute Convergence. $\boxed{A.C}$

b. Ratio Test $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{2^{n+1}(n+1)^2}}{\frac{n!}{2^n n^2}} \right| = \lim_{n\to\infty} \frac{\overbrace{(n+1)!}^{(n+1)\,\cancel{n!}}}{\cancel{n!}} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{\cancel{n^2}}{(\cancel{n}+1)^2} = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1$

$\Rightarrow$ Series $\boxed{\text{Diverges by Ratio Test}}$

(could also make a size argument to decide terms don't go to zero
$\Rightarrow$ Diverges by n^{th} term Divergence test)

c. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{n}}{n+2}$ Absolute series $\displaystyle\sum \frac{\sqrt{n}}{n+2} \approx \sum \frac{1}{\sqrt{n}}$ which is divergent p-series $p=\frac{1}{2}<1$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+2}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{n+2} = 1 \quad (0<1<\infty).$$

$\Rightarrow \displaystyle\sum \frac{\sqrt{n}}{n+2}$ also Diverges by L.C.T

Conv. by AST ① $b_n = \dfrac{\sqrt{n}}{n+2}$

② $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{1} = 0$ ✓ $\quad \frac{\infty}{\infty}$

③ $f(x) = \dfrac{\sqrt{x}}{x+2}$ decreasing?

$$f'(x) = \frac{(x+2)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}\,(1)}{(x+2)^2}$$

$$= \frac{1}{2}\sqrt{x} + \frac{1}{\sqrt{x}} - \sqrt{x}$$

$$= \frac{-\frac{1}{2}\sqrt{x} + \frac{1}{\sqrt{x}}}{(x+2)^2} < 0$$ because numerator $-\frac{1}{2}\sqrt{x} + \frac{1}{\sqrt{x}} < 0$ because

⊕ here

$\dfrac{1}{\sqrt{x}} < \dfrac{1}{2}\sqrt{x}$ when $2<x$ o.k. ✓

Therefore, original series is C.C.

$\boxed{\text{Conditionally Conv.}}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$ Ratio Test $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)}}{\frac{(x+2)^n}{2^n \ln n}} \right| = \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(x+2)^n} \right| \cdot \frac{2^n}{2^{n+1}} \cdot \underbrace{\frac{\ln n}{\ln(n+1)}}_{\to 1}$

$\displaystyle\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = 1$

$= \dfrac{|x+2|}{2}$ get convergence by Ratio Test if < 1

$\Rightarrow |x+2| < 2$

$-2 < x+2 < 2$

Endpoints $\qquad -4 < x < 0$

- $x = 0 \rightarrow$ original series becomes $\displaystyle\sum_{n=2}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n}\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ larger than $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ Divergent Harmonic ~~p-series~~ $p=1$

  $1 \le \ln n \le n \Rightarrow 1 \ge \dfrac{1}{\ln n} \ge \dfrac{1}{n}$

  Since $\dfrac{1}{\ln n} \ge \dfrac{1}{n}$ and $\sum \frac{1}{n}$ Div. then $\sum \frac{1}{\ln n}$ Diverges by CT.

- $x = -4 \rightarrow$ original series becomes

$$\sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n \cancel{2^n}}{\cancel{2^n}\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

We've already tested the Absolute Series Diverges, so that won't help us here

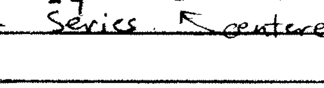
Try AST. ① $b_n = \dfrac{1}{\ln n} > 0$

② $\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$

③ $\dfrac{1}{\ln(n+1)} < \dfrac{1}{\ln n}$ since $\ln n$ is increasing function then $\dfrac{1}{\ln n}$ is decreasing.

$\Rightarrow$ converges by AST

Finally, $I = [-4, 0)$ and $R = 2$



Power Series centered at $x = -2$.

Derivatives $\quad a=1$

10 $f(x)=\frac{1}{x}$ $\quad f(1)=1=(-1)^0\,0!$

$f'(x)=-1/x^2$ $\quad f'(1)=-1=(-1)^1\,1!$

$f''(x)=\frac{2}{x^3}$ $\quad f''(1)=2=(-1)^2\,2!$

$f'''(x)=-6/x^4$ $\quad f'''(1)=-6=(-1)^3\,3!$

$f^{(4)}(x)=\frac{24}{x^5}$ $\quad f^{(4)}(1)=24=(-1)^4\,4!$

$\vdots \qquad \vdots$

In general $f^{(n)}(1)=(-1)^n n!$

$\Rightarrow$ Taylor Series $=\sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n=\sum_{n=0}^{\infty}\frac{(-1)^n \cancel{n!}}{\cancel{n!}}(x-1)^n=\boxed{\sum_{n=0}^{\infty}(-1)^n(x-1)^n}$

11. a. $\sum_{k=1}^{n}\left[\frac{k-1}{2k-1}-\frac{k}{2k+1}\right]=\underbrace{\left(0-\cancel{\frac{1}{3}}\right)}_{k=1}+\underbrace{\left(\cancel{\frac{1}{3}}-\cancel{\frac{2}{5}}\right)}_{k=2}+\underbrace{\left(\cancel{\frac{2}{5}}-\frac{3}{7}\right)}_{k=3}+\cdots+\left(\cancel{\frac{n-1}{2n-1}}-\frac{n}{2n+1}\right)$

left with last piece

$S_n=n^{th}$ Partial Sum $=\frac{-n}{2n+1}$

b. Finally the full sum is limit of the Partial Sums

$$\sum_{k=1}^{\infty}\frac{k-1}{2k-1}-\frac{k}{2k+1}=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{-n}{2n+1}=\boxed{-1/2}$$

12. Estimate $\int_0^{1/2} \frac{\ln(1+x)}{x}\,dx$ with error less than $1/100$

First $\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx = \int \sum_{n=0}^{\infty} (-x)^n\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ For $|x|<1$ of course.

Then $\frac{\ln(1+x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)\,x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

$$\int_0^{1/2} \frac{\ln(1+x)}{x}\,dx = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}\Bigg|_0^{1/2}$$

$$= \underset{n=0}{x} - \underset{n=1}{\frac{x^2}{4}} + \underset{n=2}{\frac{x^3}{9}} - \underset{n=3}{\frac{x^4}{16}} + \cdots \Bigg|_0^{1/2}$$

$$= \left(\frac{1}{2} - \frac{1}{16} + \frac{1}{72} - \frac{1}{256} + \cdots\right) - \left(0 + \cdots\right)$$

$$\begin{array}{r} 16 \\ \times 16 \\ \hline 96 \\ 160 \\ \hline 256 \end{array}$$

$\approx \boxed{\frac{1}{2} - \frac{1}{16} + \frac{1}{72}}$

If we approximate the actual (alternating) sum with only the first three terms, we will have error at most the absolute value of the 4th term (by Alternating Series Estimation Theorem) and $\frac{1}{256} < \frac{1}{100}$ our maximum desired error.