Math 12 Final Exam

May 14, 2009

Instructions: There are 12 questions on this exam for a total of 100 points. You may not use any outside materials (e.g., notes, calculators, cell phones, etc.). You have 3 hours to complete this exam. Remember to fully justify your answers.

1. (12 points) Find the following limits:

   (a) $\lim_{x\to 3} \dfrac{\ln(x-2)}{x^2-3x}$

   (b) $\lim_{x\to 0} \dfrac{\sinh x}{x}$

   (c) $\lim_{x\to 0} x\cot x$

2. (12 points) Evaluate the following integrals:

   (a) $\displaystyle\int \frac{x+3}{\sqrt{9-x^2}}\,dx.$

   (b) $\displaystyle\int \frac{dx}{x^3+x^2-2x}.$

   (c) $\displaystyle\int x\sec^2 x\,dx.$

3. (8 points) For each of the following improper integrals, determine whether it converges or diverges, and if it converges, find its value.

   (a) $\displaystyle\int_1^\infty \frac{dx}{x^2-2x+5}.$

   (b) $\displaystyle\int_0^9 \frac{dx}{(x-1)^{4/3}}.$

4. (8 points) Let R be the region bounded by the curves $y = x^2$ and $y = x + 2$.

   (a) Set up (but *don't* evaluate) an integral for the volume of the solid obtained by rotating R about the x-axis.

   (b) Set up (but *don't* evaluate) an integral for the volume of the solid obtained by rotating R about the line $x = 2$.

5. (8 points) Consider the curve given by $x = \sin^3 t$ and $y = \cos^3 t$ from $t = 0$ to $t = \frac{\pi}{2}$.

   (a) Find the tangent line to the curve at $\left(\frac{3\sqrt{3}}{8}, \frac{1}{8}\right)$.

   (b) Find the length of the curve.

6. (6 points) Let C_1 be the curve given by the polar coordinates equation $r = 2\sin\theta$, $0 \le \theta \le \pi$, and let C_2 be the curve given by the polar coordinates equation $r = 1$. Find the area of the region inside C_1 and outside C_2.

7. (6 points) Find the area of the surface obtained when the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ for $1 \le x \le 2$ is rotated about the y-axis.

8. (12 points) Determine whether each series converges absolutely, converges conditionally, or diverges. Justify your answers.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n+10)}{n^2+10n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n n^2}$.

   (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{n}}{n+2}$.

9. (8 points) Find the interval of convergence of the power series $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$.

10. (6 points) Find the Taylor series for the function $f(x) = \dfrac{1}{x}$ centered at $a = 1$. Give your answer in Σ-notation.

11. (6 points)

    (a) Find a formula for the finite sum $\displaystyle\sum_{k=1}^{n} \left[\frac{k-1}{2k-1} - \frac{k}{2k+1}\right]$. (Hint: Write out a few terms.)

    (b) Find $\displaystyle\sum_{k=1}^{\infty} \left[\frac{k-1}{2k-1} - \frac{k}{2k+1}\right]$.

12. (8 points) Use power series to estimate $\displaystyle\int_0^{1/2} \frac{\ln(1+x)}{x} dx$ with error less than $1/100$.