Math 12 Final Exam, May 11, 2011

ANSWER KEY

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to 0} \frac{1-\cosh(2x)}{x+\ln(1-x)} \overset{\frac{0}{0}\text{ L'H}}{=} \lim_{x\to 0} \frac{-2\sinh(2x)}{1-\frac{1}{1-x}} \overset{\frac{0}{0}\text{ L'H}}{=} \lim_{x\to 0} \frac{-4\cosh(2x)}{-\frac{1}{(1-x)^2}} = \frac{-4}{-1} = \boxed{4}$$

(b) $$\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{x\,1^\infty} = \lim_{x\to\infty} e^{\ln\left(\left(\frac{x}{x+1}\right)^x\right)} = e^{\lim_{x\to\infty} \ln\left(\left(\frac{x}{x+1}\right)^x\right)} = e^{\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)^{\infty\cdot 0}}$$

$$e^{\lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right)^{\frac{0}{0}}}{\frac{1}{x}}} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\left(\frac{1}{\left(\frac{x}{x+1}\right)}\right)\frac{(x+1)(1)-x(1)}{(x+1)^2}}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty} \frac{\left(\frac{x+1}{x}\right)\frac{1}{(x+1)^2}}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x\to\infty} \frac{\left(\frac{1}{x(x+1)}\right)}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty}\left(\frac{-x^2}{x(x+1)}\right)} = e^{\lim_{x\to\infty}\left(\frac{-x}{x+1}\right)} = e^{-1} = \boxed{\frac{1}{e}}$$

2. [30 Points] Evaluate each of the following **integrals**.

(a) $$\int \frac{1}{(x^2+4)^{\frac{5}{2}}}dx = \int \frac{1}{(4\tan^2\theta+4)^{\frac{5}{2}}} \cdot 2\sec^2\theta\, d\theta = \int \frac{1}{(4\sec^2\theta)^{\frac{5}{2}}} \cdot 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{(\sqrt{4\sec^2\theta})^5} \cdot 2\sec^2\theta\, d\theta = \int \frac{1}{(2\sec\theta)^5} \cdot 2\sec^2\theta\, d\theta$$

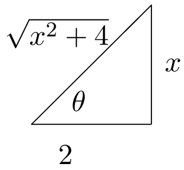
$$= \frac{1}{2^4}\int \frac{\sec^2\theta}{\sec^5\theta}\, d\theta = \frac{1}{2^4}\int \frac{1}{\sec^3\theta}\, d\theta = \frac{1}{16}\int \cos^3\theta\, d\theta$$

$$= \frac{1}{16}\int \cos^2\theta\cos\theta\, d\theta = \frac{1}{16}\int (1-\sin^2\theta)\cos\theta\, d\theta = \frac{1}{16}\int (1-w^2)\, dw$$

$$= \frac{1}{16}\left(w - \frac{w^3}{3}\right) + C = \frac{1}{16}\left(\sin\theta - \frac{\sin^3\theta}{3}\right) + C = \frac{1}{16}\left(\frac{x}{\sqrt{x^2+4}} - \frac{1}{3}\left(\frac{x}{\sqrt{x^2+4}}\right)^3\right) + C$$

$$= \boxed{\frac{1}{16}\left(\frac{x}{\sqrt{x^2+4}} - \frac{x^3}{3(x^2+4)^{\frac{3}{2}}}\right) + C}$$

Trig. Substitute $\boxed{\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta d\theta \end{array}}$



Standard w substitution for *odd* trig. integral $\int \cos^3\theta \; d\theta$ technique:

$\boxed{\begin{array}{l} w = \sin\theta \\ dw = \cos\theta \; d\theta \end{array}}$

(b) $$\int x \;\arcsin x \; dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\; dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta \; d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cdot\cos\theta \; d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta \; d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta d\theta$$

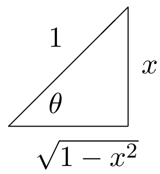
$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2} d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1-\cos(2\theta) d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}2\sin\theta\cos\theta + C = \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

$\boxed{\begin{array}{ll} u = \arcsin x & dv = xdx \\ du = \dfrac{1}{\sqrt{1-x^2}}dx & v = \dfrac{x^2}{2} \end{array}}$

Trig. Substitute $\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta d\theta \end{array}}$



(c) $$\int \frac{x^4+2x^3+7x^2+8x+7}{x^3+x^2+4x+4}\; dx = \int \frac{x^4+2x^3+7x^2+8x+7}{(x+1)(x^2+4)}\; dx$$

$$= \int x+1+\frac{2x^2+3}{x^2+4}\; dx = \int x+1+\frac{1}{x+1}+\frac{x-1}{x^2+4}\; dx$$

$$= \int x+1+\frac{1}{x+1}+\frac{x}{x^2+1}-\frac{1}{x^2+4}\; dx = \boxed{\frac{x^2}{2}+x+\ln|x+1|+\frac{\ln|x^2+4|}{2}-\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C}$$

Long division yields:

$$
\begin{array}{r}
x+1 \\
x^3+x^2+4x+4 \overline{)x^4+2x^3+7x^2+8x+7} \\
-(x^4+x^3+4x^2+4x) \\
\hline
x^3+3x^2+4x+7 \\
-(x^3+x^2+4x+4) \\
\hline
2x^2+3
\end{array}
$$

Partial Fractions Decomposition:

$$\frac{2x^2+3}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

Clearing the denominator yields:

$2x^2+3 = A(x^2+9)+(Bx+C)(x+1)$
$2x^2+3 = Ax^2+9A+Bx^2+Cx+Bx+C)$
$2x^2+3 = (A+B)x^2+(B+C)x+4A+C$
so that $A+B=2,\ B+C=0$ and $4A+C=3$
Solve for $A=1,\ B=1$ and $C=-1$

3. [20 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_7^\infty \frac{1}{x^2-6x+25}\,dx = \lim_{t\to\infty}\int_7^t \frac{1}{x^2-6x+25}\,dx = \lim_{t\to\infty}\int_7^t \frac{1}{(x-3)^2+16}\,dx$ complete the square

Substitute $\boxed{\begin{array}{l} u=x-3 \\ du=dx \end{array}}$ $\boxed{\begin{array}{l} x=7\Rightarrow u=4 \\ x=t\Rightarrow u=t-3 \end{array}}$

$$= \lim_{t\to\infty}\int_4^{t-3}\frac{1}{u^2+16}\,du = \lim_{t\to\infty}\frac{1}{16}\int_4^{t-3}\frac{1}{\left(\frac{u}{4}\right)^2+1}\,du = \lim_{t\to\infty}\frac{4}{16}\int_1^{\frac{t-3}{4}}\frac{1}{v^2+1}\,dv$$

Substitute $\boxed{\begin{array}{l} v=\frac{u}{4} \\ dv=\frac{1}{4}du \\ 4dv=du \end{array}}$ $\boxed{\begin{array}{l} u=4\Rightarrow v=1 \\ u=t-3\Rightarrow v=\frac{t-3}{4} \end{array}}$

$$= \lim_{t\to\infty}\frac{1}{4}\arctan v\Big|_1^{\frac{t-3}{4}} = \lim_{t\to\infty}\frac{1}{4}\left(\arctan\left(\frac{t-3}{4}\right)-\arctan 1\right) = \frac{1}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{1}{4}\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{16}}$$

OR you could skip all the substitution steps and go straight to

$$\lim_{t\to\infty}\int_7^t\frac{1}{(x-3)^2+16}\,dx = \lim_{t\to\infty}\frac{1}{4}\left(\arctan\left(\frac{x-3}{4}\right)\right)\Big|_7^t = \dots$$

using the formula $\displaystyle\int\frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$

(b) $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx = \lim_{s\to 3}\int_0^s \frac{1}{\sqrt{9-x^2}}\,dx = \lim_{s\to 3}\ \arcsin\left(\frac{x}{3}\right)\Big|_0^s = \lim_{s\to 3}\arcsin\left(\frac{s}{3}\right) - \arcsin(1)$

$= \dfrac{\pi}{2} - 0 = \boxed{\dfrac{\pi}{2}}$

4. [10 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4^{n+1}}{3^{2n-1}} = -\frac{4^2}{3} + \frac{4^3}{3^3} - \frac{4^4}{3^5} + \ldots$

Here we have a nice geometric series with $a = -\dfrac{16}{3}$ and $r = -\dfrac{4}{9}$

As a result, the sum is given by $\dfrac{a}{1-r} = \dfrac{-\dfrac{-16}{3}}{1-\left(-\dfrac{4}{9}\right)} = \dfrac{-\dfrac{16}{3}}{\dfrac{13}{9}} = -\dfrac{16}{3}\cdot\dfrac{9}{13} = \boxed{-\dfrac{48}{13}}$

(b) $1 - \ln 7 + \dfrac{(\ln 7)^2}{2!} - \dfrac{(\ln 7)^3}{3!} + \dfrac{(\ln 7)^4}{4!} - \dfrac{(\ln 7)^5}{5!} + \ldots = e^{(-\ln 7)} = \boxed{\dfrac{1}{7}}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^n (2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n(2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!} = \pi\cos\left(\frac{\pi}{3}\right) = \pi\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{2}}$

5. [20 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{3n^2+1}$

First, we show the absolute series is divergent. Note that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3n^2+1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent p-series with $p = 1$. Next,

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n}{3n^2+1}}{\frac{1}{n}} = \frac{n^2}{3n^2+1} = \frac{1}{3}$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, then $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3n+1}$ is also divergent by Limit Comparison Test. As a result, we have no chance for Absolute Convergence.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

$\bullet b_n = \dfrac{n}{3n^2+1} > 0$

$\bullet \lim\limits_{n\to\infty} \dfrac{n}{3n^2+1} = 0$ because the related function $f(x) = \dfrac{x}{3x^2+1}$ has negative derivative $f'(x) =$

$\bullet \dfrac{1}{b_{n+1}} < \dfrac{1}{b_n}$

$\dfrac{-3x^2+1}{(3x^2+1)^2} < 0$ when $3x^2 > 1$.

Therefore, the original series converges by the Alternating Series Test. Finally, we can conclude the original series is $\boxed{\text{Conditionally Convergent}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+4^n}{5^{n+1}} = \sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}} + \sum_{n=1}^{\infty} \frac{4^n}{5^{n+1}} = \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n + \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$

Each piece is a constant multiple of a convergent geometric series, the first one $|r| = \dfrac{3}{5} < 1$ and the second one has $|r| = \dfrac{4}{5} < 1$. Therefore, the orginal series is convergent, as a sum of two convergent series.

OR we can bound the terms of the original series

$\dfrac{3^n+4^n}{5^{n+1}} < \dfrac{4^n+4^n}{5^{n+1}} = \dfrac{2}{5}\left(\dfrac{4}{5}\right)^n$

Note $\displaystyle\frac{2}{5}\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$ is convergent, since it's a constant multiple of a convergent geometric series $|r| = \dfrac{4}{5} < 1$. Finally the orginal series is convergent by CT since we bounded the terms and the larger series converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n + n^2\sqrt{n}}{n^7+1}$

There are a couple ways to work this series.

First we can split it up into two convergent pieces.

$$\sum_{n=1}^{\infty} \frac{\arctan n + n^2\sqrt{n}}{n^7+1} = \sum_{n=1}^{\infty} \frac{\arctan n}{n^7+1} + \sum_{n=1}^{\infty} \frac{n^2\sqrt{n}}{n^7+1}$$

We can bound the terms for the first piece $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^7+1}$

$\dfrac{\arctan n}{n^7+1} < \dfrac{\frac{\pi}{2}}{n^7+1} < \dfrac{\pi}{2}\dfrac{1}{n^7}$

and $\displaystyle\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^7}$ is convergent since it's a constant multiple of a convergent p-series, $p = 7 > 1$.

Finally, since the terms are bounded, and the larger series converges, the smaller series $\sum_{n=1}^{\infty} \frac{\arctan n}{n^7+1}$ converges by CT.

We can bound the terms for the second piece $\sum_{n=1}^{\infty} \frac{n^{\frac{5}{2}}}{n^7+1}$

$$\frac{n^{\frac{5}{2}}}{n^7+1} < \frac{n^{\frac{5}{2}}}{n^7+1} < \frac{1}{n^{\frac{9}{2}}}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{2}}}$ is a convergent p-series, $p = \frac{9}{2} > 1$. Finally, since the terms are bounded, and the larger series converges, the smaller series $\sum_{n=1}^{\infty} \frac{n^2\sqrt{n}}{n^7+1}$ converges by CT.

Finally, since each piece is convergent, the sum of them is also convergent.

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OR

we can bound $\frac{\arctan n + n^2\sqrt{n}}{n^7+1} < \frac{\frac{\pi}{2} + n^2\sqrt{n}}{n^7+1}$.

If we show $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2} + n^2\sqrt{n}}{n^7+1}$ converges, then the orginal series converges by CT.

We see that $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2} + n^2\sqrt{n}}{n^7+1} \approx \sum_{n=1}^{\infty} \frac{n^2\sqrt{n}}{n^7} = \sum_{n=1}^{\infty} \frac{n^{\frac{5}{2}}}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{2}}}$ which is a convergent p-series $p = \frac{9}{2} > 1$.

Check: $\lim_{n\to\infty} \frac{\frac{\frac{\pi}{2} + n^2\sqrt{n}}{n^7+1}}{\frac{1}{n^{\frac{9}{2}}}} = \lim_{n\to\infty} \frac{\frac{\pi}{2}n^{\frac{9}{2}} + n^7}{n^7+1} = 1$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{2}}}$ converges, then the series also $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2} + n^2\sqrt{n}}{n^7+1}$ also converges by LCT.

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(d) $\sum_{n=1}^{\infty} \frac{(-1)^n \ (3n)!}{n^n \ 2^n \ \ (n!)^2 \ \ln n}$

Try Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(3(n+1))!}{(n+1)^{n+1}2^{n+1}((n+1)!)^2 \ln(n+1)}}{\frac{(-1)^n(3n)!}{n^n 2^n (n!)^2 \ln n}} \right|$$

$$= \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{(-1)^n}\right| \cdot \frac{(3n+3)!}{(3n)!} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{(n!)^2}{((n+1)!)^2} \cdot \frac{\ln n}{\ln(n+1)}$$
$$= \lim_{n\to\infty} \cdot \frac{(3n+3)(3n+2)(3n+1)(3n)!}{(3n)!} \cdot \frac{n^n}{(n+1)(n+1)^n} \cdot \frac{1}{2} \cdot \frac{(n!)^2}{(n+1)^2(n!)^2} \cdot \frac{\ln n}{\ln(n+1)}$$
$$= \lim_{n\to\infty} \cdot \frac{(3n+3)(3n+2)(3n+1)}{1} \cdot \frac{1}{(n+1)} \cdot \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{2} \cdot \frac{1}{(n+1)^2} \cdot \frac{\ln n}{\ln(n+1)}$$
$$= \lim_{n\to\infty} \cdot \frac{(3n+3)(3n+2)(3n+1)}{2(n+1)^3} \cdot \left(\frac{n}{n+1}\right)^n \cdot \frac{\ln n}{\ln(n+1)} = \frac{27}{2e} > 1$$

Here we used

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

and

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \overset{\frac{0}{0}\ L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} \lim_{n\to\infty} \frac{n+1}{n} = 1$$

The series is $\boxed{\text{Divergent by the Ratio Test}}$.

6. [10 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2x-3)^n}{n\ 6^{n+1}}$$

Use Ratio Test.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}\dfrac{(2x-3)^{n+1}}{(n+1)6^{n+2}}}{(-1)^n\dfrac{(2x-3)^n}{n6^{n+1}}} \right| = \lim_{n\to\infty} \left|\frac{(2x-3)^{n+1}}{(2x-3)^n}\right| \cdot \frac{n}{n+1} \cdot \frac{6^{n+1}}{6^{n+2}} = \frac{|2x-3|}{6}$$

The Ratio Test gives convergence for x when $\frac{|2x-3|}{6} < 1$ or $|2x-3| < 6$.

That is $-6 < 2x-3 < 6 \Longrightarrow -3 < 2x < 9 \Longrightarrow -\frac{3}{2} < x < \frac{9}{2}$

Endpoints:

•$x = \frac{9}{2}$ The original series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n\left(2\left(\frac{9}{2}\right)-3\right)^n}{n\ 6^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n 6^n}{n\ 6^{n+1}} = \frac{1}{6}\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ which is a constant multiple of the convergent alternating harmonic series. We check using AST here:

1. $b_n = \frac{1}{n} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$
3. $b_{n+1} < b_n$ since $\frac{1}{n+1} < \frac{1}{n}$

$\bullet x = -\dfrac{3}{2}$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(2\left(-\frac{3}{2}\right) - 3\right)^n}{n\ 6^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n(-6)^n}{n\ 6^{n+1}} = \frac{1}{6}\sum_{n=0}^{\infty} \frac{1}{n}$ which is a constant multiple of the divergent harmonic series $p = 1$, and therefore it's also divergent.

Finally, Interval of Convergence $\boxed{I = \left(-\frac{3}{2}, \frac{9}{2}\right]}$ with Radius of Convergence $\boxed{R = 6}$.

7. [5 Points] Write the MacLaurin Series for $\boldsymbol{f(x) = e^{-x^2}}$. Use this series to determine the **fourth** and **fifth** derivatives of $f(x) = e^{-x^2}$ at $x = 0$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

In general, the MacLaurin Series for any f is given as

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 + \dots$$

Match coefficients of like degreed terms:

$f^{(5)}(0) = \boxed{0}$ since the is no x^5 term.

$$\frac{f^{(4)}(0)}{4!} = \frac{1}{2!} \Rightarrow f^{(4)}(0) = \frac{4!}{2!} = \boxed{12}$$

8. [10 Points] Please analyze with detail and justify carefully.

(a) Find the **MacLaurin series** representation for $\boldsymbol{f(x) = x \arctan x}$.

Your answer should be in sigma notation $\displaystyle\sum_{n=0}^{\infty}$.

$$\arctan x = \int \frac{1}{1+x^2}\ dx \int \frac{1}{1-(-x^2)}\ dx = \int \sum_{n=0}^{\infty} \left(-x^2\right)^n = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

OR Memorize

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$x \arctan x = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1}$$

(b) Use the MacLaurin series representation for $\boldsymbol{f(x) = x \arctan x}$ from Part(a) to

Estimate $\displaystyle\int_0^{\frac{1}{2}} x \arctan\ dx$ with error less than $\dfrac{1}{100}$.

Justify in words that your error is indeed less than $\frac{1}{100}$.

$$\int_0^{\frac{1}{2}} x \arctan\ dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1}\ dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+3)(2n+1)}\Bigg|_0^{\frac{1}{2}} = \frac{x^3}{3} - \frac{x^5}{5\cdot 3} + \frac{x^7}{7\cdot 5} + \ldots\Bigg|_0^{\frac{1}{2}}$$

$$= \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^5}{15} + \frac{\left(\frac{1}{2}\right)^7}{35} + \ldots = \left(\frac{1}{24} - \frac{1}{480} + \ldots\right) - (0 - 0 + 0 + \ldots) \approx= \boxed{\frac{1}{24}}$$

Note this is an alternating series. Use the Alternating Series Estimation Theorem. If we approximate the actual sum with only the first term, the error from the actual sum will be at most the absolute value of the next term, $\frac{1}{480}$. Here $\frac{1}{480} < \frac{1}{100}$ as desired.

9. [15 Points] Consider the region bounded by $\boldsymbol{y = \cos x}$, $\boldsymbol{y = \sin x}$, $\boldsymbol{x = 0}$ and $\boldsymbol{x = \frac{\pi}{4}}$. Rotate the region about the $\boxed{\boldsymbol{y}\text{–axis}}$. **Compute** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

See me for a sketch.

$\boldsymbol{V = \int_0^{\frac{\pi}{4}} 2\pi}$ radius height $\boldsymbol{dx = \int_0^{\frac{\pi}{4}} 2\pi x(\cos x - \sin x)\ dx}$ I.B.P. here (or can split into two I.B.P)

$$\boxed{\begin{array}{ll} u = x & dv = \cos x - \sin x\ dx \\ du = dx & v = \sin x + \cos x \end{array}}$$

$$= 2\pi\left(x(\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin x + \cos x\ dx\right)$$

$$= 2\pi\left(\tfrac{\pi}{4}(\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)) - (0) - (-\cos x + \sin x)\Big|_0^{\frac{\pi}{4}}\right)$$

$$= 2\pi\left(\tfrac{\pi}{4}(\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)) - (0) - \left(\left(-\cos\left(\tfrac{\pi}{4}\right) + \sin\left(\tfrac{\pi}{4}\right)\right) - (-\cos 0 + \sin 0)\right)\right)$$

$$= 2\pi\left(\tfrac{\pi}{4}(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}) - \left(\left(-\tfrac{\sqrt{2}}{2}\right) + \tfrac{\sqrt{2}}{2}\right) - (-1 + 0)\right) = \boxed{2\pi\left(\tfrac{\sqrt{2}\pi}{4} - 1\right)} = \tfrac{\sqrt{2}\pi^2}{2} - 1$$

10. [15 Points] Consider the Parametric Curve represented by $\boldsymbol{x = e^t - t}$ and $\boldsymbol{y = 4e^{t/2}}$.

(a) Compute the **arclength** of this parametric curve for $\boldsymbol{0 \le t \le 1}$.

First, $\boldsymbol{\frac{dx}{dt} = e^t - 1}$ and $\boldsymbol{\frac{dy}{dt} = 2e^{t/2}}$.

$$\boldsymbol{L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^1 \sqrt{(e^t - 1)^2 + \left(2e^{t/2}\right)^2}\ dt = \int_0^1 \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\ dt}$$

$$= \int_0^1 \sqrt{e^{2t} + 2e^t + 1}\, dt = \int_0^1 \sqrt{(e^t+1)^2}\, dt$$

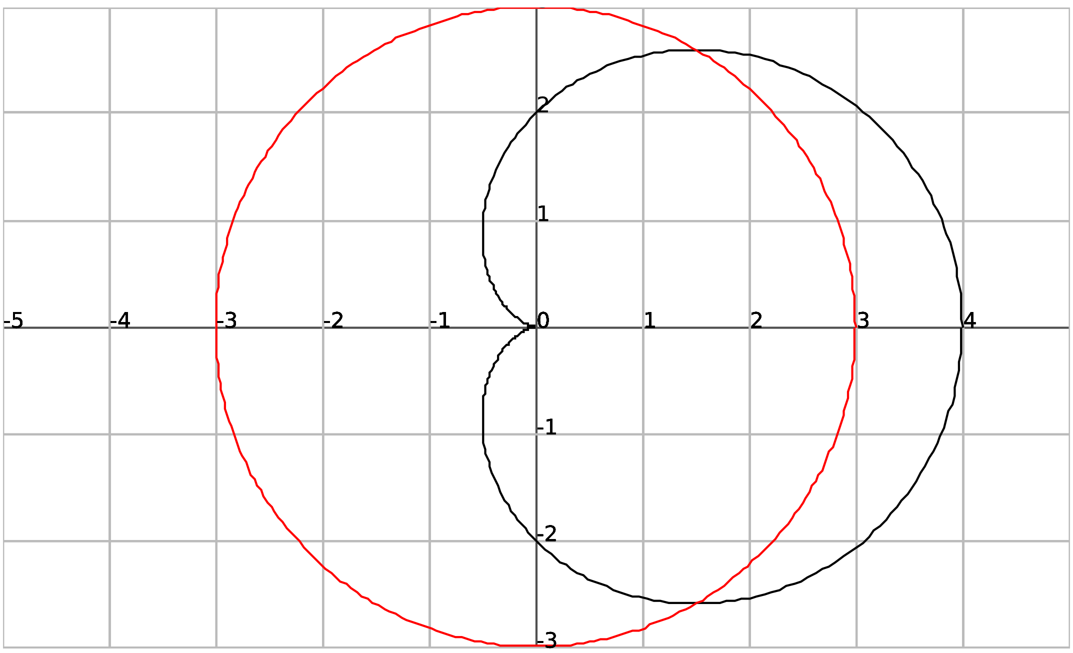
$$= \int_0^1 e^t + 1\, dt = e^t + t\Big|_0^1 = (e+1) - (e^0 + 0) = e + 1 - 1 = \boxed{e}$$

(b) Set-up, **BUT DO NOT EVALUATE**, the definite integral representing the **surface area** obtained by rotating this curve about the y**-axis**, for $0 \le t \le 1$.

$$\text{S.A.} = \int_0^1 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi x(t)\sqrt{(e^t-1)^2 + \left(2e^{t/2}\right)^2}\, dt$$

$$= \boxed{\int_0^1 2\pi\left(e^t - t\right)\left(e^t + 1\right)\, dt} \text{ from part(a) above.}$$

11. [15 Points] Compute the **area** bounded inside the polar curve $r = 2 + 2\cos\theta$ and outside the polar curve $r = 3$. **Sketch** the Polar curves.



These two polar curves intersect when $2 + 2\cos\theta = 3 \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = -\frac{\pi}{3}$ or $\theta = \frac{\pi}{3}$. Using symmetry, we will integrate from $\theta = 0$ to $\theta = \frac{\pi}{3}$ and double that area.

$$\text{Area} = A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} \left((\text{outer } r)^2 - (\text{inner } r)^2\right)\, d\theta\right) = \int_0^{\frac{\pi}{3}} \left((2 + 2\cos\theta)^2 - 3^2\right)\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} 4 + 8\cos\theta + 4\cos^2\theta - 9\, d\theta = \int_0^{\frac{\pi}{3}} -5 + 8\cos\theta + 4\left(\frac{1+\cos(2\theta)}{2}\right)\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} -5 + 8\cos\theta + 2 + 2\cos(2\theta)\, d\theta = \int_0^{\frac{\pi}{3}} -3 + 8\cos\theta + 2\cos(2\theta)\, d\theta$$

$$= -3\theta + 8\sin\theta + \sin(2\theta)\Big|_0^{\frac{\pi}{3}} = \left(-\pi + 8\sin\left(\tfrac{\pi}{3}\right) + \sin\left(\tfrac{2\pi}{3}\right)\right) - (0 + 8\sin 0 + \sin 0)$$

$$= -\pi + 8\left(\tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{\sqrt{3}}{2}\right) = -\pi + 4\sqrt{3} + \tfrac{\sqrt{3}}{2} = \boxed{\frac{9\sqrt{3}}{2} - \pi}$$

12. [10 Points] Find the general solution for each of the following **differential equations**.

(a) $\dfrac{dy}{dx} = (\ln x)\sqrt{1-y^2}$ Separable

$$\frac{1}{\sqrt{1-y^2}}\,dy = \ln x\,dx$$

Antidifferentiate: $\displaystyle\int \frac{1}{\sqrt{1-y^2}}\,dy = \int \ln x\,dx = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - \int 1\,dx$

$$\boxed{\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x}\,dx & v = x \end{array}}$$

$\arcsin y = x\ln x - x + C$ by Integration by Parts

Finally, $\boxed{y = \sin(x\ln x - x + C)}$

Note, $y = 1$ trivial solution here.

(b) $x\dfrac{dy}{dx} - y = x^2e^x$ Linear First Order

Linear Form: $\dfrac{dy}{dx} - \dfrac{1}{x}y = xe^x$

Integrating Factor: $I(x) = e^{\int P(x)\,dx} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = e^{\ln(|x|^{-1})} = \dfrac{1}{|x|}$

Take $I(x) = \dfrac{1}{x}$

Multiply Diff. Eq. in its linear form by $I(x)$:

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$$

Recognize left side as a product rule derivative:

$$\left(\frac{1}{x}y\right)' = e^x$$

Antidifferentiate: $\dfrac{1}{x}y = e^x + C$

Finally, $\boxed{\boldsymbol{y = x(e^x + C)}}$