Amherst College, DEPARTMENT OF MATHEMATICS

Math 12 Final Exam

May 13, 2010

Answer Key

1. [10 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to 3} \frac{\arctan(x-3)}{x^2-9} = \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 3} \frac{\frac{1}{(x-3)^2+1}}{2x} = \boxed{\frac{1}{6}}$$

(b) $$\lim_{x\to\infty} (e^x+1)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln\left((e^x+1)^{\frac{1}{x}}\right)} = e^{\lim_{x\to\infty} \ln\left((e^x+1)^{\frac{1}{x}}\right)} = e^{\lim_{x\to\infty} \frac{\ln(e^x+1)}{x}}$$

$$= e^{\left(\frac{\infty}{\infty}\right)} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\frac{e^x}{e^x+1}}{1}} = e^{\left(\frac{\infty}{\infty}\right)} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{e^x}{e^x}} = e^1 = \boxed{e}$$

2. [15 Points] Evaluate each of the following **integrals**.

(a) $$\int_0^{\ln 7} \frac{\sinh x}{\cosh x} dx = \int_1^{\frac{25}{7}} \frac{1}{u} du = \ln|u|\Big|_1^{\frac{25}{7}} = \ln\left(\frac{25}{7}\right) - \ln 1 = \boxed{\ln\left(\frac{25}{7}\right)}$$

Substitute $\boxed{\begin{array}{l} u = \cosh x \\ du = \sinh x dx \end{array}}$ $\boxed{\begin{array}{l} x = 0 \Rightarrow u = \cosh 0 = 1 \\ x = \ln 7 \Rightarrow u = \cosh(\ln 7) = \dfrac{e^{\ln 7} + \frac{1}{e^{\ln 7}}}{2} = \dfrac{7 + \frac{1}{7}}{2} = \dfrac{25}{7} \end{array}}$

(b) $$\int x \ \arcsin x \ dx = \frac{x^2}{2} \arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} \ dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \ d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta \ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta \ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2} d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1 - \cos(2\theta) d\theta$$

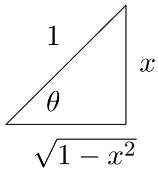
$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8} 2\sin\theta\cos\theta + C = \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

$$u = \arcsin x \qquad dv = x\,dx$$

$$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = \frac{x^2}{2}$$

Trig. Substitute $x = \sin\theta$, $dx = \cos\theta\, d\theta$



3. [20 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $$\int_7^\infty \frac{1}{x^2-8x+19}\,dx = \lim_{t\to\infty}\int_7^t \frac{1}{x^2-8x+19}\,dx = \lim_{t\to\infty}\int_7^t \frac{1}{(x-4)^2+3}\,dx$$ complete the square

Substitute $$u = x-4, \quad du = dx \qquad x = 7 \Rightarrow u = 3, \quad x = t \Rightarrow u = t-4$$

$$= \lim_{t\to\infty}\int_3^{t-4}\frac{1}{u^2+3}\,du = \lim_{t\to\infty}\frac{1}{3}\int_3^{t-4}\frac{1}{\left(\frac{u}{\sqrt{3}}\right)^2+1}\,du = \lim_{t\to\infty}\frac{\sqrt{3}}{3}\int_{\sqrt{3}}^{\frac{t-4}{\sqrt{3}}}\frac{1}{v^2+1}\,dv$$

Substitute $$v = \frac{u}{\sqrt{3}}, \quad dv = \frac{1}{\sqrt{3}}du, \quad \sqrt{3}dv = du \qquad u = 3 \Rightarrow v = \frac{3}{\sqrt{3}} = \sqrt{3}, \quad u = t-4 \Rightarrow v = \frac{t-4}{\sqrt{3}}$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan v\Big|_{\sqrt{3}}^{\frac{t-4}{\sqrt{3}}} = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-4}{\sqrt{3}}\right) - \arctan\sqrt{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right)$$

$$= \boxed{\frac{\pi}{6\sqrt{3}}}$$

(b) $$\int_0^4 \frac{3x+2}{x^2-x-12}\,dx = \int_0^4\frac{3x+2}{(x-4)(x+3)}\,dx = \lim_{t\to 4^-}\int_0^t\frac{3x+2}{(x-4)(x+3)}\,dx$$

$$= \lim_{t\to 4^-}\int_0^t \frac{2}{x-4}+\frac{1}{x+3}\,dx = \lim_{t\to 4^-} 2\ln|x-4| + \ln|x+3|\Big|_0^t$$

$$= \lim_{t\to 4^-}(2\ln|t-4| + \ln|t+3|) - (2\ln 4 + \ln 3) = 2\ln 0^+ + \ln 7 - 2\ln 4 - \ln 3 = -\infty \ \boxed{\text{Diverges}}$$

Partial Fractions Decomposition:

$$\frac{3x+2}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$$

Clearing the denominator yields:

$3x + 2 = A(x+3) + B(x-4)$
$3x + 2 = (A+B)x + (3A - 4B)$
so that $A + B = 3$ and $3A - 4B = 2$
Solve for $A = 2$ and $B = 1$

4. [10 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{9^{n-1}} = -4 + \frac{16}{9} - \frac{64}{81} + \ldots$

Here we have a nice geometric series with $a = -4$ and $r = -\frac{4}{9}$

As a result, the sum is given by $\displaystyle\frac{a}{1-r} = \frac{-4}{1-\left(-\frac{4}{9}\right)} = \frac{-4}{\frac{13}{9}} = -4 \cdot \frac{9}{13} = \boxed{-\frac{36}{13}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!} = \boxed{e^3}$

(c) $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n} \left(\frac{\pi}{6}\right)}{(2n+1)! \left(\frac{\pi}{6}\right)} = \frac{6}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n+1}}{(2n+1)!}$$
$$= \frac{6}{\pi} \sin\left(\frac{\pi}{6}\right) = \frac{6}{\pi} \cdot \frac{1}{2} = \boxed{\frac{3}{\pi}}$$

5. [20 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}$

First, we show the absolute series is divergent. Note that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent p-series with $p = 1$. Next,

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{3n+1}}{\frac{1}{n}} = \frac{n}{3n+1} = \frac{1}{3}$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$ is also divergent by Limit Comparison Test. As a result, we have no chance for Absolute Convergence.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

$\bullet b_n = \dfrac{1}{3n+1} > 0$

$\bullet \lim\limits_{n\to\infty} \dfrac{1}{3n+1} = 0$

$\bullet \dfrac{1}{b_{n+1}} < \dfrac{1}{b_n}$ because $\dfrac{1}{3(n+1)+1} < \dfrac{1}{3n+1}$ or $\dfrac{1}{3n+4} < \dfrac{1}{3n+1}$

Therefore, the original series converges by the Alternating Series Test. Finally, we can conclude the original series is $\boxed{\text{Conditionally Convergent}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+9}{n+1}$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$

since $\displaystyle\lim_{n\to\infty} \frac{e^n+9}{n+1} = \frac{\infty}{\infty} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{e^n}{1} = \infty \neq 0.$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

Try Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1}}}{\dfrac{(-1)^n n!}{n^n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \right| \cdot \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)n!}{n!} \cdot \frac{n^n}{(n+1)(n+1)^n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$$

The series is $\boxed{\text{Absolutely Convergent by the Ratio Test}}$.

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3+8}{n^7-9}$

First examine the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+8}{n^7-9} \approx \sum_{n=1}^{\infty} \frac{1}{n^4}$ which is a convergent p-series with $p = 4 > 1$.

Check: $\displaystyle\lim_{n\to\infty} \frac{\dfrac{n^3+8}{n^7-9}}{\dfrac{1}{n^4}} = \lim_{n\to\infty} \frac{n^7+8n^4}{n^7-9} = 1$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges, then the absolute series also

$\boxed{\text{Converges by Limit Comparison Test}}$. As a result, we have Absolute Convergence, which implies the original series is $\boxed{\text{Absolutely Convergent}}$.

6. [10 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(3x-1)^n}{n^2 \cdot 2^n}$$

Use Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(3x-1)^{n+1}}{(n+1)^2 2^{n+1}}}{\dfrac{(-1)^n(3x-1)^n}{n^2 2^n}} \right| = \lim_{n\to\infty} \left| \frac{(3x-1)^{n+1}}{(3x-1)^n} \right| \cdot \frac{n^2}{(n+1)^2} \cdot \frac{2^n}{2^{n+1}} = \frac{|3x-1|}{2}$$

The Ratio Test gives convergence for x when $\dfrac{|3x-1|}{2} < 1$ or $|3x-1| < 2$.

That is $-2 < 3x - 1 < 2 \Longrightarrow -1 < 3x < 3 \Longrightarrow -\dfrac{1}{3} < x < 1$

Endpoints:

•$x = 1$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n^2 2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ which is a convergent by Absolute Convergence Test since its absolute series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$.

•$x = -\dfrac{1}{3}$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{n^2 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n 2^n}{n^2 2^n} = \sum_{n=0}^{\infty} \frac{1}{n^2}$ which is a convergent p-series with $p = 2 > 1$.

Finally, Interval of Convergence $\boxed{I = \left[-\frac{1}{3}, 1\right]}$ with Radius of Convergence $\boxed{R = \frac{2}{3}}$.

7. [5 Points] Consider the function $f(x)$ that satisfies the following

$$f(3) = 2$$

$$f'(3) = -3$$

$$f''(3) = \frac{6}{7}$$

$$f'''(3) = -1$$

Find the **Taylor polynomial of degree 3** for $\mathbf{f(x)}$ centered at $\mathbf{a = 3}$.

$$
\begin{aligned}
T_3(x) &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 \\
&= 2 - 3(x-3) + \frac{\frac{6}{7}}{2!}(x-3)^2 - \frac{1}{3!}(x-3)^3 \\
&= \boxed{2 - 3(x-3) + \frac{3}{7}(x-3)^2 - \frac{1}{6}(x-3)^3}
\end{aligned}
$$

8. [10 Points] **MacLaurin Series:** Please analyze with detail and justify carefully.

(a) Find the **MacLaurin series** representation for $\boldsymbol{f(x) = x\ln(1+x)}$.

Your answer should be in sigma notation $\sum_{n=0}^{\infty}$.

First, $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

Next, $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^n}$

Then, $\ln(1+x) = \displaystyle\int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x)^n\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

Finally, $x\ln(1+x) = x \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n+1}}$

(b) Use the MacLaurin Power Series representation for $\boldsymbol{f(x) = x\ln(1+x)}$ from Part(a) to

$$
\textbf{Estimate} \quad \int_0^1 x\ln(1+x)\,dx \quad \text{with error less than } \frac{1}{10}.
$$

Now, $\displaystyle\int_0^1 x\ln(1+x)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n+1}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{(n+3)(n+1)}\Bigg|_0^1$

$$
= \frac{x^3}{3} - \frac{x^4}{4\cdot 2} + \frac{x^5}{5\cdot 3} + \ldots \Bigg|_0^1 = \left(\frac{1}{3} - \frac{1}{8} + \frac{1}{15} + \ldots\right) - (0 - 0 + 0 + \ldots) \approx \frac{1}{3} - \frac{1}{8} = \boxed{\frac{5}{24}}
$$

Note this is an alternating series. Use the Alternating Series Estimation Theorem. If we approximate the actual sum with only the first two terms, the error from the actual sum will be at most the absolute value of the next term, $\frac{1}{15}$. Here $\frac{1}{15} < \frac{1}{10}$ as desired.

9. [15 Points] Volumes of Revolution

(a) Consider the region bounded by $\boldsymbol{y = e^x}$, $\boldsymbol{y = \ln x}$, $\boldsymbol{x = 1}$, and $\boldsymbol{x = 2}$. Rotate the region about the $\boxed{\boldsymbol{y}\text{–axis}}$. **Compute** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

See me for a sketch.

$$V = \int_1^2 2\pi \text{ radius height } dx = \int_1^2 2\pi x(e^x - \ln x)\, dx = 2\pi \int_1^2 xe^x - x\ln x\, dx$$

Double Integration By Parts:

$$\boxed{\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}} \quad \boxed{\begin{array}{ll} u = \ln x & dv = x dx \\ du = \frac{1}{x} dx & v = \frac{x^2}{2} \end{array}}$$

$$= 2\pi \left(\int_1^2 xe^x\, dx - \int_1^2 x\ln x\, dx \right) = 2\pi \left(xe^x \Big|_1^2 - \int_1^2 e^x\, dx - \left(\frac{x^2}{2} \ln x \Big|_1^2 - \int_1^2 \frac{x}{2}\, dx \right) \right)$$

$$= 2\pi \left(xe^x \Big|_1^2 - e^x \Big|_1^2 - \left(\frac{x^2}{2} \ln x \Big|_1^2 - \frac{x^2}{4} \Big|_1^2 \right) \right) = 2\pi \left(2e^2 - e^1 - (e^2 - e^1) - \left(2\ln 2 - \frac{1}{2} \ln 1 - (1 - \frac{1}{4}) \right) \right)$$

$$= 2\pi \left(2e^2 - e^1 - e^2 + e^1 - 2\ln 2 + 0 + 1 - \frac{1}{4} \right) = \boxed{2\pi \left(e^2 - 2\ln 2 + \frac{3}{4} \right)}$$

(b) Consider the same region bounded by $\boldsymbol{y = e^x}$, $\boldsymbol{y = \ln x}$, $\boldsymbol{x = 1}$, and $\boldsymbol{x = 2}$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = -3}}$. Set up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method.

See me for a sketch.

$$V = \int_1^2 2\pi \text{ radius height } dx = \int_1^2 2\pi (x+3)(e^x - \ln x)\, dx$$

10. [15 Points] Consider the Parametric Curve represented by $\boldsymbol{x = e^t - t}$ and $\boldsymbol{y = 4e^{t/2}}$.

(a) Find the **arclength** of this parametric curve for $0 \leq t \leq 1$.

First, $\frac{dx}{dt} = e^t - 1$ and $\frac{dy}{dt} = 2e^{t/2}$.

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(e^t - 1)^2 + \left(2e^{t/2}\right)^2}\, dt = \int_0^1 \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt$$

$$= \int_0^1 \sqrt{e^{2t} + 2e^t + 1}\, dt = \int_0^1 \sqrt{(e^t + 1)^2}\, dt$$

$$= \int_0^1 e^t + 1\, dt = e^t + t \Big|_0^1 = (e + 1) - \left(e^0 + 0\right) = e - +1 - 1 = \boxed{e}$$

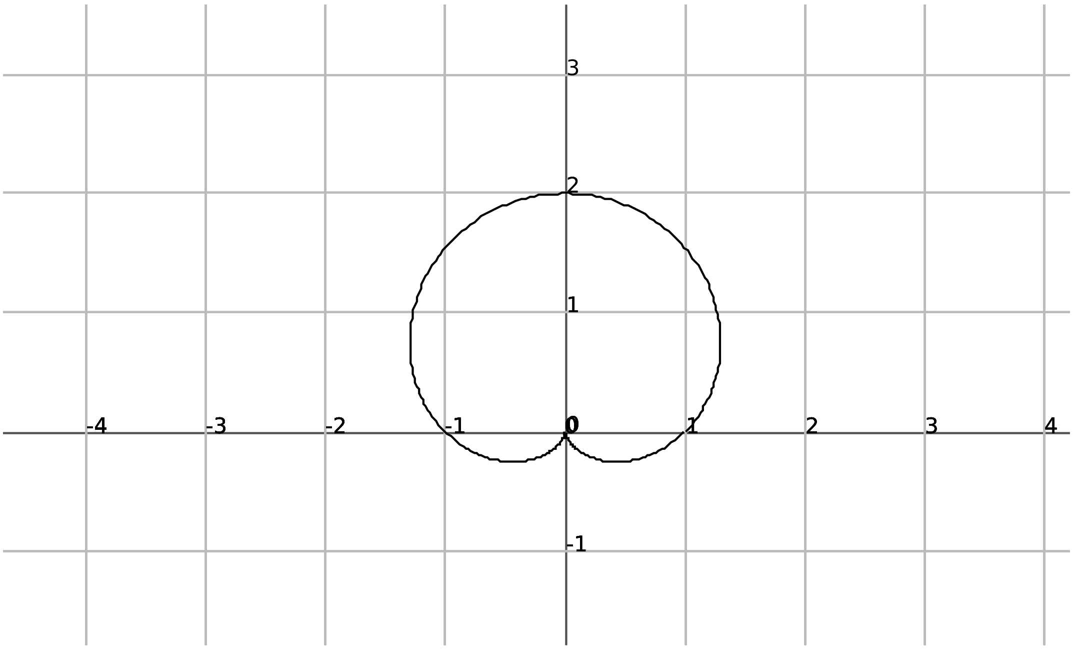
(b) Set up, **BUT DO NOT EVALUATE!!**, the definite integral representing the **surface area** of the solid obtained by rotating this curve about the $\boldsymbol{x}$-axis, for $0 \le t \le 1$.

$$\text{S.A.} = \int_0^1 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^1 2\pi y(t)\sqrt{(e^t-1)^2 + \left(2e^{t/2}\right)^2}\ dt$$

$$= \boxed{\int_0^1 2\pi\left(4e^{t/2}\right)\left(e^t+1\right)\ dt}$$ from part(a) above.

11. [15 Points] For each part, **sketch** the Polar curve(s), and answer the related questions:

(a) **Compute** the **area** enclosed by the cardioid $\boldsymbol{r = 1 + \sin\theta}$.



It's important to note that one full cycle of the cardioid closes up on itself as θ ranges from $\theta = 0$ to $\theta = 2\pi$.

$$A = \frac{1}{2}\int_0^{2\pi} r^2\ d\theta = \frac{1}{2}\int_0^{2\pi}(1+\sin(\theta))^2\ d\theta = \frac{1}{2}\int_0^{2\pi} 1 + 2\sin\theta + \sin^2\theta\ d\theta$$

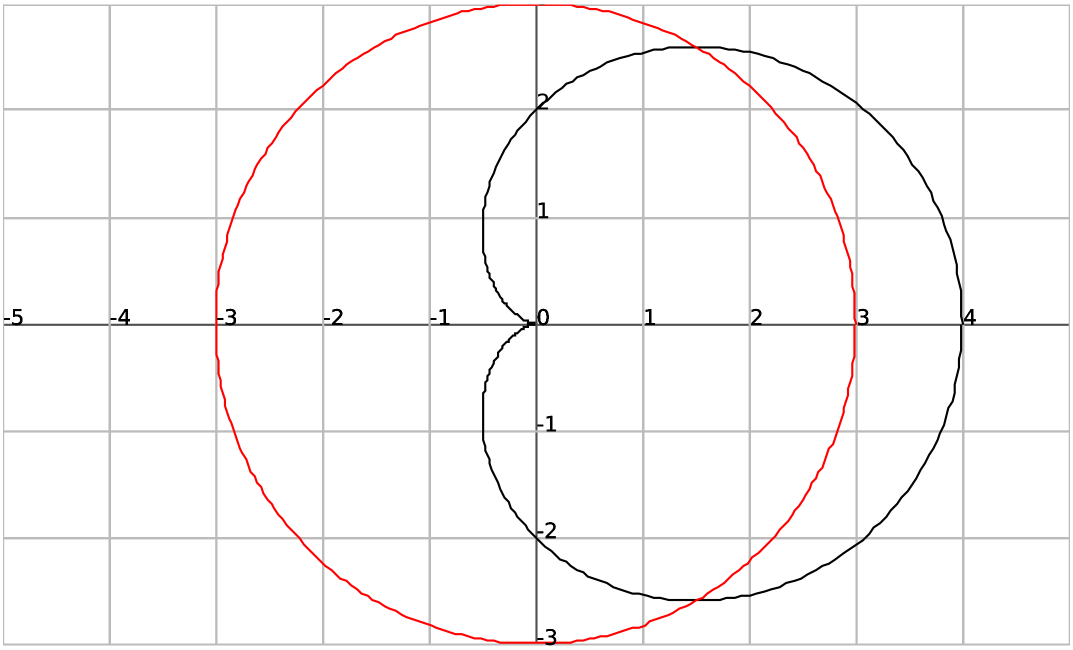
$$= \frac{1}{2}\int_0^{2\pi} 1 + 2\sin\theta + \left(\frac{1-\cos(2\theta)}{2}\right)\ d\theta = \frac{1}{2}\int_0^{2\pi} 1 + 2\sin\theta + \frac{1}{2} - \frac{\cos(2\theta)}{2}\ d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{3}{2} + 2\sin\theta - \frac{\cos(2\theta)}{2}\ d\theta = \frac{1}{2}\left(\frac{3}{2}\theta - 2\cos\theta - \frac{\sin(2\theta)}{4}\right)\Bigg|_0^{2\pi}$$

$$= \frac{1}{2}\left(\left(3\pi - 2\cos(2\pi) - \frac{\sin(4\pi)}{4}\right) - (0 - 2\cos 0 - 0)\right)$$

$$\frac{1}{2}\left((3\pi - 2 - 0) - (0 - 2 - 0)\right) = \frac{1}{2}(3\pi - 2 + 2) = \boxed{\frac{3\pi}{2}}$$

(b) Set up, **BUT DO NOT EVALUATE!!**, the definite integral representing the **area** bounded inside $\boldsymbol{r = 2 + 2\cos\theta}$ and outside $\boldsymbol{r = 3}$.



These two polar curves intersect when $2 + 2\cos\theta = 3 \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = -\frac{\pi}{3}$ or $\theta = \frac{\pi}{3}$. Using symmetry, we will integrate from $\theta = 0$ to $\theta = \frac{\pi}{3}$ and double that area.

$$\text{Area} = A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2\right)\ d\theta\right) = \int_0^{\frac{\pi}{3}} \left((2 + 2\cos\theta)^2 - 3^2\right)\ d\theta$$

12. [10 Points] Find the general solution for each of the following **differential equations**.

(a) $\dfrac{dy}{dx} = (1 + y^2)e^x$ Separable $\dfrac{1}{1 + y^2}dy = e^x dx$

Antidifferentiate: $\displaystyle\int \frac{1}{1 + y^2}\ dy = \int e^x\ dx$

$\arctan y = e^x + C$ Finally, $\boxed{y = \tan\left(e^x + C\right)}$

(b) $\dfrac{dy}{dx} + 2xy = e^{-x^2}$

(already in) Linear Form: $\dfrac{dy}{dx} + 2xy = e^{-x^2}$

Integrating Factor: $I(x) = e^{\displaystyle\int P(x)\ dx} = e^{\displaystyle\int 2x\ dx} = e^{x^2}$

Take $I(x) = e^{x^2}$.

Multiply Diff. Eq. in its linear form by $I(x)$:

$e^{x^2}\dfrac{dy}{dx} + 2xe^{x^2}y = e^{-x^2}e^{x^2} = e^0 = 1$

Recognize left side as a product rule derivative:

$\left(e^{x^2}y\right)' = 1$

Antidifferentiate:

$e^{x^2}y = x + C$

Finally, $\boxed{y = \dfrac{x + C}{e^{x^2}}}$