Math 121, Section(s) 01, Spring 2025

Homework #16

Due **Friday, April 11th** in Gradescope by 11:59 pm ET

Goal: Exploring more of the Relationship between Power Series and functions, including Differentiation and Integration of Power Series. Also *substitution* into a known MacLaurin Series. Also SUMS which are not Geometric.

FIRST: Read through and understand the following Examples. Simplify.

Ex: Use *Substitution* to find the Series for this function and find the Radius of Convergence.

$$\frac{d}{dx}\left(x^2\ln(1+5x)\right) \overset{\text{sub}}{=} \frac{d}{dx}\left(x^2\sum_{n=0}^{\infty}\frac{(-1)^n\left(5x\right)^{n+1}}{n+1}\right) = \frac{d}{dx}\left(x^2\sum_{n=0}^{\infty}\frac{(-1)^n5^{n+1}x^{n+1}}{n+1}\right)$$

$$= \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n5^{n+1}x^{n+3}}{n+1} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n5^{n+1}(n+3)x^{n+2}}{n+1}} \text{ Need } |5x|<1$$

$\Rightarrow |x| < \frac{1}{5}$. $R = \frac{1}{5}$ The Radius is unchanged after Differentiation. Here $\boxed{R=\frac{1}{5}}$ still.

Note how we were able to avoid running a Ratio Test for find the Radius of Convergence.

Ex: Find the Series Representation for $\ln(9+x^2)$. One method uses Integration.

$$\ln(9+x^2) \overset{\text{hint}}{=} \int \frac{2x}{9+x^2}dx = \int 2x\cdot\frac{1}{9+x^2}dx = \int\frac{2x}{9}\cdot\frac{1}{1-\left(-\frac{x^2}{9}\right)}dx$$

$$= \int\frac{2x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n dx = \int\frac{2x}{9}\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{9^n}dx = \int 2\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{9^{n+1}}dx$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+2}}{9^{n+1}(2n+2)} + C = 2\left(\frac{x^2}{9\cdot 2} - \frac{x^4}{9^2\cdot 4} + \frac{x^6}{9^3\cdot 6} - \dots\right) + C$$

Test $x=0$ into both sides to solve for C. Expand in long form to fully justify.

$$\ln(9+0) = 2\left(\frac{0}{9\cdot 2} - \frac{0}{9^2\cdot 4} + \frac{0}{9^3\cdot 6} - \dots\right) + C \Rightarrow C = \ln 9 \quad \text{Plug } C \text{ value back in}$$

$$\text{Finally, } \ln(9+x^2) = \boxed{2\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+2}}{9^{n+1}(2n+2)} + \ln 9} \overset{\text{or}}{=} \sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+2}}{9^{n+1}(n+1)} + \ln 9$$

Need $\left|-\frac{x^2}{9}\right| < 1 \Rightarrow |x|^2 < 9 \Rightarrow |x| < 3$ so $R=3$. The Radius remains unchanged after Integration above. So $\boxed{R=3}$ still.

Continue on and Complete the next page of problems

Find the Series Representation for the following functions using *substitution* and determine the Radius of Convergence R. Simplify.

1. $\dfrac{1}{1+x^2}$

2. $\dfrac{x^2}{x^4+16}$

3. $x^3\cos\left(x^2\right)$

4. $5x^2\sin\left(5x\right)$

5. $\dfrac{d}{dx}\left(x^3\arctan(7x)\right)$

6. $\displaystyle\int x^3\arctan(7x)\ dx$

7. $\dfrac{d}{dx}x^2\ln(1+6x)$

8. $\displaystyle\int x^4e^{-x^3}\ dx$

9. Find the Series Representation for $f(x) = \dfrac{1}{(1+x)^2}$

Hint: $\dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\cancelto{PS?}{-\dfrac{1}{1+x}}\right) = \dots$

10. Prove the Power Series Representation formula for $\arctan x$, as shown in class. Yes, show that $C=0$.

It is most convincing to expand your series in *long expanded form* to best solve for C, by plugging in or testing the value of the center point, here $x=0$, into both sides of the equation

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + C$$

11. Find Series Representation for $\ln(5-x)$. Solve for C and the Radius R.

Hint: $\ln(5-x) = \displaystyle\int \frac{-1}{5-x}\ dx = \int \frac{-1}{5\left(1-\frac{x}{5}\right)}\ dx = -\frac{1}{5}\int \cancelto{PS?}{\frac{1}{1-\frac{x}{5}}}\ dx = \dots$

12. Find the MacLaurin Series for $f(x) = e^{-2x}$ using two different methods. Your answers should be in Sigma notation.

First, using the *Definition* of the MacLaurin Series ("Chart Method").

Second, use *Substitution* into a known series.

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13. You do **not** need to state the Radius. Answers should be in Sigma notation $\sum_{n=0}^{\infty}$ here.

You may use the fact that $\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$ without extra justification.

(a) Use the Definition ("Chart Method") to compute the MacLaurin Series for $F(x) = \cos x$.

(b) Use Differentiation to compute the Series for $F(x) = \cos x$.

$$\text{Hint: } \cos x = \frac{d}{dx} \sin x = \frac{d}{dx} \overset{\text{PS?}}{\cancel{\sin x}} = \dots$$

(c) Use Integration to compute the Series for $F(x) = \cos x$.

$$\text{Hint: } \cos x = \int -\sin x \; dx = \int -\overset{\text{PS?}}{\cancel{\sin x}} \; dx = \dots$$

Hints: yes, you should solve for $+C$. yes, C should equal 1. Show why $C = 1$.

Find the Sum of each of the following Series, which do converge.

14. $\sum_{n=0}^{\infty} \frac{7^n}{n!}$

15. $\sum_{n=0}^{\infty} \frac{(-1)^n \, 5^n}{n!}$

16. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$

17. $\sum_{n=0}^{\infty} \frac{(-1)^n \, \pi^{2n}}{6^{2n} \, (2n)!}$

18. $\sum_{n=0}^{\infty} \frac{3^n}{5^n \, n!}$

19. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

20. $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \dots$

21. $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \dots$

REGULAR OFFICE HOURS

Monday: 12:00–3:00 pm

6:00–7:30 pm TA Jude, SMUDD 207

Tuesday: 1:00–4:00 pm

6–7:30 pm TA Aaron, SMUDD 207

Wednesday: 1:00-3:00 pm

6–7:30 pm TA Jude, SMUDD 207

7:30–9:00 pm TA Aaron, SMUDD 207

Thursday: none for Professor

extras may be added, TBD weekly

7:30–9:00 pm TA DJ, SMUDD 207

Friday: 12:00–3:00 pm

7:30–9:00 pm TA DJ, SMUDD 207

Pay careful attention to details here.
Manipulating power series requires a balance
of memory and technical skill.