Math 121 Exam 3
April 25, 2025

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

- Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{3\ln 3}$, or $\arctan\sqrt{3}$ should be simplified.

- Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [16 Points] Interval/Radius of Convergence Analyze carefully, with full justification.

Find the **Interval** and **Radius** of Convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x+1)^n}{(n+5)\cdot 7^n}$

2. [22 Points] Find the Series for each of the functions. Also **STATE** the Radius of Convergence for each series. Answers should be in sigma notation $\displaystyle\sum_{n=0}^{\infty}$. Simplify.

(a) $\displaystyle\frac{x^2}{4+x}\,dx \overset{\text{hint}}{=} x^2\left(\frac{1}{4+x}\right)$ (b) $\displaystyle\frac{d}{dx}\left(5x^4\arctan(5x)\right)$ (c) $\displaystyle\int x^5\cos(6x^2)\,dx$

3. [12 Points] Use Series to **Estimate** $\displaystyle\int_0^1 x^3\sin\left(x^2\right)\,dx$ with error less than $\frac{1}{1000}$.

Simplify. Justify. Tip: $(5!)\cdot(14) = 1680$

4. [26 Points] Find the **sum** for each of the following convergent series. Simplify.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(36)^n\ (2n+1)!}$ (b) $3-\dfrac{3}{2}+1-\dfrac{3}{4}+\dfrac{3}{5}-\dfrac{1}{2}+\ldots$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ \pi^{2n+1}}{4!\ (2n)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 8)^n}{3^n \cdot n!}$ (e) $-1+\dfrac{1}{3}-\dfrac{1}{5}+\dfrac{1}{7}-\dfrac{1}{9}+\ldots$ (f) $1+1-\dfrac{\pi^2}{2!}+\dfrac{\pi^4}{4!}-\dfrac{\pi^6}{6!}+\dfrac{\pi^8}{8!}-\ldots$

5. [12 Points] Use Series to compute the following Limit $\displaystyle\lim_{x\to 0} \frac{x-\ln(1+x)}{e^{3x}-1-3x}$

6. [12 Points] Find the MacLaurin Series Representation for $\arctan\left(x^3\right)$.
You **MUST** use Integration of Series, and the following Hint formula.

Hint: $\boxed{\arctan\left(x^3\right) = \displaystyle\int \overset{\nearrow\text{Power Series}}{\frac{3x^2}{1+x^6}}\,dx = \int 3x^2\left(\frac{1}{1+x^6}\right)\,dx}$ Yes, solve for C

OPTIONAL: Feel free to check your answer using a *substitution*.

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Create a limit $\displaystyle\lim_{x\to 0}\frac{G(x)}{H(x)}$ which equals $\boxed{-\dfrac{1}{16}}$ and requires 3 applications of L'Hôpital's Rule. The expression $\dfrac{G(x)}{H(x)}$ must include at least two of $\sin x$, $\ln(1+x)$, $\cos x$, $\arctan x$, and e^x. Continue on to compute the Limit using Series, and then also using the three L'H Rules.