Exam 3 Spring 25 Answer Key

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(3x+1)^n}{(n+5)\cdot 7^n}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(3x+1)^{n+1}}{(n+6)\,7^{n+1}}}{\frac{\cancel{(-1)^n}(3x+1)^n}{(n+5)\cdot 7^n}}\right| = \lim_{n\to\infty}\left|\frac{(3x+1)^{n+1}}{(3x+1)^n}\right|\cancel{\left(\frac{n+5}{n+6}\right)}^{1}\cdot\frac{\cancel{7^n}}{7^{n+1}}$$

$|3x+1|$

$7^{n+1} = \cancel{7^n}\cdot 7$

$$= \lim_{n\to\infty}\frac{|3x+1|}{7} = \frac{|3x+1|}{7} < 1$$

Converges by the Ratio Test when

$$|3x+1| < 7 \Rightarrow \begin{array}{c} -7 < 3x+1 < 7 \\ -1 \qquad\qquad -1 \\ -8 < 3x < 6 \end{array} \Rightarrow -\frac{8}{3} < x < 2$$

Manually Test Convergence at Endpoints

Take $x = 2$, Series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(3(2)+1)^n}{(n+5)\cdot 7^n} = \sum_{n=0}^{\infty}\frac{(-1)^n\cancel{7^n}}{(n+5)\cancel{7^n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+5}$

Converges by the Alternating Series Test

1. $b_n = \frac{1}{n+5} > 0$
2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\cancel{\frac{1}{n+5}}^{\,0} = 0$ $\quad(n+5 \to \infty)$
3. Terms Decreasing

   $b_{n+1} = \frac{1}{n+6} \leq \frac{1}{n+5} = b_n$

Take $x = -\frac{8}{3}$, Series becomes

$(-1)^{2n}$ even $\to 1$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\left(3\left(\frac{-8}{3}\right)+1\right)^n}{(n+5)\cdot 7^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(-7)^n}{(n+5)\,7^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n\cancel{7^n}}{(n+5)\cdot\cancel{7^n}} = \sum_{n=0}^{\infty}\frac{1}{n+5} \approx \sum_{n=0}^{\infty}\frac{1}{n}$$

Divergent

p-Series $p=1$

LCT Limit: $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{n+5}}{\frac{1}{n}} = \lim_{n\to\infty}\cancel{\frac{n}{n+5}}^{\,1} = 1$ Finite Non-Zero $\Rightarrow$ Series also Diverges by LCT

Finally, $\boxed{I = \left(-\frac{8}{3}, 2\right] \quad R = \frac{7}{3}}$

$2 - \left(-\frac{8}{3}\right) = \frac{6}{3} + \frac{8}{3} = \frac{14}{3} \xrightarrow{\text{half}} \frac{7}{3}$

Total Length

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } |x|<1$$

2(a) $$\frac{x^2}{4+x} = x^2\left(\frac{1}{4+x}\right) = \frac{x^2}{4}\left(\frac{1}{1+\frac{x}{4}}\right) = \frac{x^2}{4}\left(\frac{1}{1-\left(-\frac{x}{4}\right)}\right)$$

$$= \frac{x^2}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n = \frac{x^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{4^{n+1}}}$$

Need $\left|-\frac{x}{4}\right|<1$

$|x|<4 \Rightarrow R=4$

2(b) $$\frac{d}{dx}\left(5x^4\arctan(5x)\right) = \frac{d}{dx}\left(5x^4\sum_{n=0}^{\infty}\frac{(-1)^n(5x)^{2n+1}}{2n+1}\right)$$

Need $|5x|<1$

$\Rightarrow |x|<\frac{1}{5}$ $R=\frac{1}{5}$

$$= \frac{d}{dx}\left(5x^4\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}x^{2n+1}}{2n+1}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+2}x^{2n+5}}{2n+1}\right)$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+2}(2n+5)x^{2n+4}}{2n+1}}$$

$R=\frac{1}{5}$ STILL After Differentiation

2(c) $$\int x^5\cos(6x^2)\,dx = \int x^5\sum_{n=0}^{\infty}\frac{(-1)^n(6x^2)^{2n}}{(2n)!}\,dx = \int x^5\sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n}x^{4n}}{(2n)!}\,dx$$

$R=\infty$

$$= \int\sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n}x^{4n+5}}{(2n)!}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n}x^{4n+6}}{(2n)!\,(4n+6)} + C$$

$R=\infty$ STILL

After Integration

$$3 \quad \int_0^1 x^3 \sin(x^2)\,dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}\,dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\,dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+6}}{(2n+1)!(4n+6)} \Bigg|_0^1$$

$n=0 \quad n=1 \quad n=2 \quad \dots$

$$= \frac{x^6}{1\cdot 6} - \frac{x^{10}}{3!\cdot(10)} + \frac{x^{14}}{\underbrace{5!\,(14)}_{1680}} - \dots \Bigg|_0^1$$

Tip

Full Sum

Show Zeroes

$$= \frac{1}{6} - \frac{1}{60} + \frac{1}{1680} - \dots - \cancel{(0 - 0 + 0 - \dots)}^{\,0}$$

$$\approx \frac{1}{6} - \frac{1}{60} = \frac{10}{60} - \frac{1}{60} = \frac{9}{60} = \frac{3}{20} \leftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem (A.S.E.T.) we can Estimate the full Sum using only the first two terms with Error at most $\frac{1}{1680} < \frac{1}{1000}$ as desired

$$4(a) \quad \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{\underbrace{(36)^n}_{6^{2n}}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!} \frac{\frac{\pi}{6}}{\frac{\pi}{6}} = \frac{1}{\frac{\pi}{6}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n+1}}{(2n+1)!}$$

extra

$$= \frac{6}{\pi} \cancel{\sin\left(\frac{\pi}{6}\right)}^{\frac{1}{2}} = \frac{6}{\pi} \cdot \frac{1}{2} = \frac{3}{\pi}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4(b) $3-\frac{3}{2}+1-\frac{3}{4}+\frac{3}{5}-\frac{1}{2}+\ldots = 3-\frac{3}{2}+\frac{3}{3}-\frac{3}{4}+\frac{3}{5}-\frac{3}{6}+\ldots$

$$= 3\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\right)$$

$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$

$$= 3\ln(1+1) = 3\ln 2$$

4(c) $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n+1}}{4!\,(2n)!} = -\frac{\pi}{4!}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!} = -\frac{\pi}{24}\overset{-1}{\cancel{\cos\pi}} = \frac{\pi}{24}$

$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

4(d) $\sum_{n=0}^{\infty}\frac{(-1)^n(\ln 8)^n}{3^n\cdot n!} = \sum_{n=0}^{\infty}\frac{\left(\frac{-\ln 8}{3}\right)^n}{n!} = e^{\frac{-\ln 8}{3}} = e^{-\frac{1}{3}\ln 8}$

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$= \cancel{e}^{\ln\left(8^{-1/3}\right)} = 8^{-\frac{1}{3}} = \frac{1}{8^{1/3}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$$

4(e) $-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\ldots = -\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\right)$

$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$

$$= -\overset{\frac{\pi}{4}}{\cancel{\arctan 1}} = -\frac{\pi}{4}$$

Note: $\arctan(-1) = -\frac{\pi}{4}$ also works here

4(f) $1+1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\frac{\pi^8}{8!}-\ldots = 1+\overset{-1}{\cancel{\cos\pi}} = 0$

$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

5. $$\lim_{x\to 0} \frac{x-\ln(1+x)}{e^{3x}-1-3x} = \lim_{x\to 0} \frac{\cancel{x}-\left(\cancel{x}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)}{\left(\cancel{1}+\cancel{3x}+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\cdots\right)-\cancel{1}-\cancel{3x}}$$

$\frac{0}{0}$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\cdots}{\frac{9x^2}{2!}+\frac{3^3x^3}{3!}+\frac{3^4x^4}{4!}+\cdots} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to 0} \frac{\frac{1}{2}-\frac{x}{3}+\frac{x^2}{4}-\cdots}{\frac{9}{2}+\frac{3^3x}{3!}+\frac{3^4x^2}{4!}+\cdots}$$

$$= \frac{\frac{1}{2}}{\frac{9}{2}} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{9} = \frac{1}{9}$$

Can check using L'H Rule Two times : Optional

Geometric Sub

6. $$\arctan(x^3) = \int \frac{3x^2}{1+x^6}\,dx = \int 3x^2\left(\frac{1}{1-(-x^6)}\right)dx = \int 3x^2 \sum_{n=0}^{\infty}(-x^6)^n\,dx$$

$$= \int 3x^2 \sum_{n=0}^{\infty}(-1)^n x^{6n}\,dx = \int 3\sum_{n=0}^{\infty}(-1)^n x^{6n+2}\,dx$$

$n=0 \quad n=1 \quad n=2 \ldots$

$$= 3\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{6n+3} + C = 3\left(\frac{x^3}{3}-\frac{x^9}{9}+\frac{x^{15}}{15}-\ldots\right)+C$$

Test $x=0$

$$\cancel{\arctan 0} = 3(0-0+0-\ldots)+C \Rightarrow C=0$$

Finally, $$\arctan(x^3) = \boxed{\cancel{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{6n+3}} \text{ OR } = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{2n+1}}$$

$\cancel{3}(2n+1)$

Optional: using Substitution

$$\arctan(x^3) = \sum_{n=0}^{\infty}\frac{(-1)^n (x^3)^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{2n+1}}$$ Match!

Optional Bonus: Create $\displaystyle\lim_{x\to 0}\frac{G(x)}{H(x)} = -\frac{1}{16}$

$$\lim_{x\to 0}\frac{x-\sin x}{\arctan(2x)-2x}\ \tfrac{0}{0} \underset{L'H}{=} \lim_{x\to 0}\frac{1-\cos x}{\frac{2}{1+(2x)^2}-2}\ \tfrac{0}{0}$$

$2(1+4x^2)^{-1}$

$\searrow$

$-2(1+4x^2)^{-2}(8x)$

$$\underset{L'H}{=} \lim_{x\to 0}\frac{+\sin x}{\frac{-16x}{(1+4x^2)^2}}\ \tfrac{0}{0}$$

Quotient Rule

$$\underset{L'H}{=} \lim_{x\to 0}\frac{\cancel{\cos x}\to 1}{\frac{\cancel{(1+4x^2)^2}\to 1\,(-16)-\cancel{(-16x)2(1+4x^2)(8x)}\to 0}{(1+4x^2)^4\to 1}}$$

yuck!

$$=\frac{1}{-16} = -\frac{1}{16}$$

$$\lim_{x\to 0}\frac{x-\sin x}{\arctan(2x)-2x} = \lim_{x\to 0}\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)}{2x-\frac{(2x)^3}{3}+\frac{(2x)^5}{5}-\dots-2x}$$

$$=\lim_{x\to 0}\frac{\cancel{x}-\cancel{x}+\frac{x^3}{3!}-\frac{x^5}{5!}+\dots}{\cancel{2x}-\frac{2^3x^3}{3}+\frac{2^5x^5}{5}-\dots-\cancel{2x}}$$

$$=\lim_{x\to 0}\frac{\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\dots}{-\frac{2^3x^3}{3}+\frac{2^5x^5}{5}-\frac{2^7x^7}{7}+\dots}\quad \begin{matrix}\frac{1}{x^3}\\[4pt] \frac{1}{x^3}\end{matrix}$$

$$=\lim_{x\to 0}\frac{\frac{1}{3!}-\cancel{\frac{x^2}{5!}+\frac{x^4}{7!}}\to 0-\dots}{-\frac{8}{3}+\cancel{\frac{2^5x^2}{5}-\frac{2^7x^4}{7}}\to 0+\dots} = \frac{\frac{1}{3!}}{-\frac{8}{3}}\ \ \tfrac{-3}{8} = \frac{-1}{2\cdot 8} = \frac{-1}{16}$$

Yes! Match!

OR $\lim_{x\to 0} \dfrac{x\cos x - x}{48\left(e^x - 1 - x - \frac{x^2}{2!}\right)}$

helps scale

$$\cancel{x} - \frac{x^3}{2!} + \frac{x^4}{4!} - \dots \cancel{-x}$$

$$= \lim_{x\to 0} \frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - x}{48\left(\cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \cancel{-1} \cancel{-x} \cancel{-\frac{x^2}{2!}}\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{x^3}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}{48\left(\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right)} \quad \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2!} + \frac{x}{4!} - \frac{x^3}{6!} + \dots}{48\left(\frac{1}{3!} + \frac{x}{4!} + \frac{x^2}{5!} + \dots\right)} = \frac{-\frac{1}{2}}{48\left(\frac{1}{6}\right)} = \frac{-\frac{1}{2}}{8} = -\frac{1}{16}$$

$\frac{1}{8}$

$$\lim_{x\to 0} \frac{x\cos x - x}{48\left(e^x - 1 - x - \frac{x^2}{2!}\right)} \quad \frac{0}{0} \quad \underset{\text{L'H}}{=} \lim_{x\to 0} \frac{x(-\sin x) + \cos x\,(1) - 1}{48\left(e^x - 1 - x\right)} \quad \frac{0}{0}$$

$-2\sin x$

$$\underset{\text{L'H}}{=} \lim_{x\to 0} \frac{x(-\cos x) + (-\sin x)\cdot 1 - \sin x}{48\left(e^x - 1\right)} \quad \frac{0}{0}$$

$$\underset{\text{L'H}}{=} \lim_{x\to 0} \frac{x\underbrace{(\sin x)}_{\to 0} + \underbrace{(-\cos x)}_{\to -1}(1) - 2\underbrace{\cos x}_{\to -1}}{48\underbrace{e^x}_{\to 1}}$$

$$= \frac{-3}{48} = -\frac{1}{16}$$

Yes! Match!

OR

$$\lim_{x\to 0}\frac{\ln(1+x)-x+\frac{x^2}{2}}{4\left(\sin(2x)-2x\right)} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{\frac{1}{1+x}-1+x}{4\left(2\cos(2x)-2\right)} \quad \frac{0}{0}$$

helps scale

$$\underset{\text{L'H}}{=} \lim_{x\to 0}\frac{\frac{-1}{(1+x)^2}+1}{-16\sin(2x)} \quad \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{\frac{2}{(1+x)^3} \to 2}{-32\cos(2x) \to 1}$$

$$= \frac{2}{-32} = -\frac{1}{16}$$

$$\lim_{x\to 0}\frac{\ln(1+x)-x+\frac{x^2}{2}}{4\left(\sin(2x)-2x\right)} = \lim_{x\to 0}\frac{\cancel{x}-\cancel{\frac{x^2}{2}}+\frac{x^3}{3}-\frac{x^4}{4}+\dots-\cancel{x}+\cancel{\frac{x^2}{2}}}{4\left(\cancel{2x}-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\dots-\cancel{2x}\right)}$$

$$= \lim_{x\to 0}\frac{\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots}{4\left(-\frac{8x^3}{3!}+\frac{2^5x^5}{5!}-\frac{2^7x^7}{7!}+\dots\right)} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to 0}\frac{\frac{1}{3}-\frac{x}{4}+\frac{x^2}{5}-\dots \to 0}{4\left(\frac{-8}{3!}+\frac{2^5x^2}{5!}-\frac{2^7x^4}{7!}+\dots\right) \to 0}$$

$$= \frac{\frac{1}{3}}{\frac{-32}{6}} = \frac{1}{\cancel{3}}\cdot\frac{\cancel{6}^{\,2}}{-32} = \frac{-2}{32} = -\frac{1}{16}$$

Yes! Match!