Exam 2 Spring 25 Answer Key

1(a) $\displaystyle\int_0^e x^4 \cdot \ln x \, dx = \lim_{t\to 0^+} \int_t^e x^4 \cdot \ln x \, dx = \lim_{t\to 0^+} \frac{x^5}{5} \cdot \ln x \Big|_t^e - \frac{1}{5}\int_t^e x^4 \, dx$

$$\boxed{\begin{matrix} u = \ln x & dv = x^4 \, dx \\ du = \frac{1}{x} dx & v = \frac{x^5}{5} \end{matrix}}$$

$$= \lim_{t\to 0^+} \frac{x^5}{5} \cdot \ln x \Big|_t^e - \frac{x^5}{25}\Big|_t^e$$

$$= \lim_{t\to 0^+} \frac{e^5}{5} \cdot \overset{1}{\cancel{\ln e}} - \overset{0\cdot(-\infty) \to 0}{\cancel{\frac{t^5}{5} \ln t}} - \left(\frac{e^5}{25} - \overset{\to 0}{\cancel{\frac{t^5}{25}}}\right)$$

☆ see below

$$= \frac{e^5}{5} - \frac{e^5}{25} = \frac{5e^5}{25} - \frac{e^5}{25} = \frac{4e^5}{25} \quad \text{Converges}$$

$$☆ \lim_{t\to 0^+} \overset{0\cdot(-\infty)}{t^5 \cdot \ln t} = \lim_{t\to 0^+} \frac{\ln t}{1/t^5}\,\overset{-\infty}{\underset{\infty}{}} \underset{\text{L'H}}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-5}{t^6}} \cdot \frac{t^6}{-5} = \lim_{t\to 0^+} \frac{\overset{\to 0}{\cancel{-t^5}}}{5} = 0$$

$t^{-5} \to -5t^{-6}$

1(b) $\displaystyle\int_{-\infty}^1 \frac{1}{x^2-6x+13} dx = \lim_{t\to -\infty} \int_t^1 \frac{1}{x^2-6x+13} dx = \lim_{t\to-\infty} \int_t^1 \frac{1}{(x-3)^2+4} dx$

Complete the Square $\quad x^2-6x+9+\underline{4}$

$$\boxed{\begin{matrix} u = x-3 \\ du = dx \end{matrix}}$$

$$\boxed{\begin{matrix} x = t \Rightarrow u = x-3 \\ x = 1 \Rightarrow u = -2 \end{matrix}}$$

$$= \lim_{t\to-\infty} \int_{t-3}^{-2} \frac{1}{u^2+4} du = \lim_{t\to-\infty} \frac{1}{2} \arctan\left(\frac{u}{2}\right)\Big|_{t-3}^{-2}$$

$$= \lim_{t\to-\infty} \frac{1}{2}\left(\overset{-\frac{\pi}{4}}{\cancel{\arctan\left(\overset{-1}{\cancel{\tfrac{-2}{2}}}\right)}} - \overset{-\frac{\pi}{2}}{\cancel{\arctan\left(\overset{-\infty}{\cancel{\tfrac{t-3}{2}}}\right)}}\right)$$

$$= \frac{1}{2}\left(-\frac{\pi}{4} + \underbrace{\frac{\pi}{2}}_{\frac{2\pi}{4}}\right) = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8} \quad \text{Converges}$$

$$1(c)\quad \int_{-7}^{0} \frac{x+15}{x^2+6x-7}\,dx = \lim_{t\to -7^+} \int_t^0 \frac{x+15}{x^2+6x-7}\,dx = \lim_{t\to -7^+} \int_t^0 \frac{x+15}{(x-1)(x+7)}\,dx$$

PFD

$$\cancel{(x+7)(x-1)}\left[\frac{x+15}{\cancel{(x-1)(x+7)}} = \frac{A}{x-1} + \frac{B}{x+7}\right](x+7)(x-1)$$

$$x+15 = A(x+7) + B(x-1)$$
$$= Ax + 7A + Bx - B$$
$$= (A+B)x + (7A-B)$$

Conditions

- $A+B=1 \Rightarrow B = 1-A$
- $7A - B = 15$

$$7A-(1-A)=15$$
$$7A-1+A=15$$
$$8A=16 \qquad B=1-2=-1$$
$$A=2$$

PFD

$$= \lim_{t\to -7^+} \int_t^0 \frac{2}{x-1} - \frac{1}{x+7}\,dx$$

$$= \lim_{t\to -7^+} 2\ln|x-1| - \ln|x+7| \Big|_t^0$$

$$= \lim_{t\to -7^+} 2\cancel{\ln|-1|}^{\,\ln 1 \to 0} - \underbrace{\ln 7}_{\text{Finite}} - \Big(\underbrace{\cancel{2\ln|t-1|}}_{\text{Finite}}^{\,2\ln 8} - \cancel{\ln|t+7|}^{\,-\infty}\Big)$$

$t \to -7^+$, $t+7 \to 0^+$

$$= -(-(-\infty)) = -\infty \qquad \text{Diverges}$$

$$2.\quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \quad \xrightarrow{\text{Not Equal}} \quad \int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2}\,du$$

$u^{-2} \to \frac{u^{-1}}{-1}$

u-sub

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = 2 &\Rightarrow u = \ln 2 \\ x = t &\Rightarrow u = \ln t \end{aligned}}$$

$$= \lim_{t\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} -\cancel{\frac{1}{\ln t}}^{\,0} + \frac{1}{\ln 2}$$

$\ln t \to \infty$

$$= \frac{1}{\ln 2} \quad \text{Integral Converges}$$

$$\Rightarrow \boxed{\text{Original Series Converges by the Integral Test}}$$

Important: Two Separate Conclusions

$$3(a)\quad \sum_{n=2}^{\infty} \frac{n^6}{\ln n} \quad \longrightarrow \quad \boxed{\text{Diverges by nTDT}} \text{ because}$$

$$\lim_{n\to\infty} \frac{n^6}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{x^6}{\ln x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{6x^5}{\frac{1}{x}} = \lim_{x\to\infty} 6\cancel{x^6}^{\,\infty} = \infty \neq 0$$

3(b) $-4-\frac{4}{2}-\frac{4}{3}-1-\frac{4}{5}-\frac{4}{6}-\ldots = -4\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\right)$

$-\frac{4}{4}$

$$= -4\sum_{n=1}^{\infty}\frac{1}{n}$$

Constant Multiple of the Divergent (Harmonic) p-Series $p=1$ is $\boxed{\text{Divergent}}$

3(c) $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^6}+\frac{(-6)^n}{7^{2n}} \overset{\text{split}}{=} \sum_{n=1}^{\infty}\frac{4}{n^6}+\sum_{n=1}^{\infty}\frac{(-6)^n}{7^{2n}}$

$$=4\sum_{n=1}^{\infty}\frac{1}{n^6}+\sum_{n=1}^{\infty}\frac{(-6)^n}{49^n}$$

$r=\frac{-6}{49}$

$\frac{-6}{49}+\frac{6^2}{(49)^2}-\frac{6^3}{(49)^3}+\ldots$

Constant Multiple of a Convergent p-Series $p=6>1$ is Convergent

Convergent by GST with $|r|=\left|\frac{-6}{49}\right|=\frac{6}{49}<1$

Sum of Two Convergent Series is $\boxed{\text{Convergent}}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4+6} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty}\frac{1}{n^4+6} \approx \sum_{n=1}^{\infty}\frac{1}{n^4}$ Convergent p-Series $p=4>1$

Bound Terms

$$\frac{1}{n^4+6}\le\frac{1}{n^4}$$

The Absolute Series also Converges by CT

ACT

The Original Series also Converges by the Absolute Convergence Test

OR LCT works on the Absolute Series

$$5(a)\ \sum_{n=1}^{\infty}(-1)^n\left(\frac{n^4+6}{n^6+4}\right) \xrightarrow{A.S.} \sum_{n=1}^{\infty}\frac{n^4+6}{n^6+4} \approx \sum_{n=1}^{\infty}\frac{n^4}{n^6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Convergent p-Series $p=2>1$

don't need to study the original series

LCT Limit

$$\lim_{n\to\infty}\frac{\frac{n^4+6}{n^6+4}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^6+6n^2}{n^6+4}\cdot\frac{\frac{1}{n^6}}{\frac{1}{n^6}} = \lim_{n\to\infty}\frac{1+\frac{6}{n^4}\to 0}{1+\frac{4}{n^6}\to 0} = 1$$

Finite Non-Zero

$\Rightarrow$ the Absolute Series also Converges by LCT

$\Rightarrow$ the Original Series is $\boxed{\text{Absolutely Convergent}}$ (by Definition)

$$5(b)\ \sum_{n=1}^{\infty}\frac{(-1)^n n^n (2n)!}{6^n (n!)^3}$$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(n+1)^{n+1}(2(n+1))!}{6^{n+1}((n+1)!)^3}}{\frac{\cancel{(-1)^n} n^n (2n)!}{6^n (n!)^3}}\right| \quad \frac{6^n(n!)^3}{n^n(2n)!}$$

$$= \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\cdot\frac{(2n+2)!}{\cancel{(2n)!}}\cdot\frac{\cancel{6^n}}{6^{n+1}}\cdot\frac{\cancel{(n!)^3}}{((n+1)!)^3}$$

$(n+1)^{n+1} = (n+1)^n(n+1)$

$(2n+2)! = (2n+2)(2n+1)\cancel{(2n)!}$

$6^{n+1} = \cancel{6^n}\cdot 6$

$((n+1)!)^3 = ((n+1)n!)^3 = (n+1)^3\cancel{(n!)^3}$

3 copies

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n}\cdot\frac{1}{6}\cdot\left(\frac{\cancel{n+1}}{\cancel{n+1}}\right)\left(\frac{2n+2}{\cancel{n+1}}\right)\left(\frac{2n+1}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$\frac{(n+1)^n}{n^n}\to e$, $\quad 1$, $\quad 2n+2 = 2\cancel{(n+1)}$

$$= \lim_{n\to\infty}\frac{2e}{6}\left(\frac{2+\frac{1}{n}\to 0}{1+\frac{1}{n}\to 0}\right) = \frac{4e}{6} = \frac{2e}{3} > 1$$

$\boxed{\text{Original Series Diverges by Ratio Test}}$

$2e>4$

$e\approx 2.7$

5(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+4} \xrightarrow[\text{A.S.}]{\text{1st}} \sum_{n=1}^{\infty} \frac{1}{6n+4} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Divergent (Harmonic) p-Series $p=1$

2nd AST

1. Isolate $b_n = \frac{1}{6n+4} > 0$

2. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{\cancel{6n+4}} = 0$

∞, 0

3. Terms Decreasing

$b_{n+1} = \frac{1}{\underbrace{6(n+1)+4}_{6n+10}} \leq \frac{1}{6n+4} = b_n$

Original Series Converges by AST

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{1}{6n+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{6n+4} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{6+\cancel{\frac{4}{n}}} = \frac{1}{6}$$

Finite + Non-Zero

$\Rightarrow$ Absolute Series Diverges by LCT

Original Series is $\boxed{\text{Conditionally Convergent}}$ by Definition

Bonus:

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$$

First: Terms approach 0 as $n\to\infty$ $\quad \lim\limits_{n\to\infty} \cancel{\ln\left(1+\cancel{\frac{1}{n}}\right)} = 0$ ✓

Second: Use the Integral Test to show the Series Diverges

$$\int_1^{\infty} \ln\left(1+\frac{1}{x}\right) dx = \lim_{t\to\infty} \int_1^t \underbrace{\ln\left(1+\frac{1}{x}\right)}_{L} \cdot \underbrace{1}_{P}\, dx$$

IBP

$$\boxed{\begin{array}{ll} u = \ln\left(1+\frac{1}{x}\right) & dv = 1\,dx \\ du = \frac{1}{1+1/x}\left(-\frac{1}{x^2}\right)dx & v = x \end{array}}$$

$\underbrace{}_{\frac{-1}{x^2+x}}$

$$= \lim_{t\to\infty} x \cdot \ln\left(1+\frac{1}{x}\right)\Big|_1^t \; \overset{--}{+} \int_1^t \frac{1}{x+1}\,dx$$

Bonus (Continued)

$$= \lim_{t\to\infty} x\cdot \ln\left(1+\frac{1}{x}\right)\Big|_1^t + \ln|x+1|\Big|_1^t$$

$$= \lim_{t\to\infty} \overset{\infty\cdot 0}{t\cdot \ln\left(1+\frac{1}{t}\right)} \xrightarrow{\ 1\ } - \underbrace{1\cdot \ln 2}_{\text{Finite}} + \ln|\cancel{t+1}| \xrightarrow{\ \infty\ } - \underbrace{\ln 2}_{\text{Finite}}$$

☆ Finite

$= \infty$ Integral Diverges, therefore the $\boxed{\text{Series Diverges by Integral Test}}$

$$☆ \lim_{t\to\infty} t\cdot \ln\left(1+\frac{1}{t}\right) = \lim_{t\to\infty} \frac{\ln\left(1+\frac{1}{t}\right)}{\frac{1}{t}} \overset{0/0}{\underset{L'H}{=}} \lim_{t\to\infty} \frac{\frac{1}{1+1/t}\left(\cancel{-\frac{1}{t^2}}\right)}{\cancel{-\frac{1}{t^2}}} = \lim_{t\to\infty} \frac{1}{1+\frac{1}{t}} = 1$$

Alternate ?

Note: Can also use Algebra

$$\int_1^\infty \ln\left(1+\frac{1}{x}\right)dx = \int_1^\infty \ln\left(\frac{x+1}{x}\right)dx = \int_1^\infty \ln|x+1| - \ln x \; dx$$

and run two separate IBPs

but may require more work

on the Indeterminate Product

Limit Finishes