Math 121 Exam 2 Spring 2024 Answer Key

1(a) $\int_0^e x^2\ln(x^2)\,dx = \lim_{t\to 0^+}\int_t^e x^2\ln(x^2)\,dx = \lim_{t\to 0^+} \frac{x^3}{3}\cdot\ln(x^2)\Big|_t^e - \frac{2}{3}\int_t^e x^2\,dx$

$u=\ln(x^2)$	$dv = x^2\,dx$
$du=\frac{1}{x^2}(2x)\,dx = \frac{2}{x}\,dx$	$v=\frac{x^3}{3}$

$= \lim_{t\to 0^+} \frac{x^3}{3}\cdot\ln(x^2)\Big|_t^e - \frac{2x^3}{9}\Big|_t^e$

$= \lim_{t\to 0^+} \frac{e^3}{3}\cdot\ln(e^2) - \frac{t^3}{3}\cdot\ln(t^2) - \left(\frac{2e^3}{9} - \frac{2t^3}{9}\right)$

$0\cdot(-\infty) \to 0$ See (*); $\ln(e^2) = 2$; $\frac{2t^3}{9} \to 0$

$= \frac{2e^3}{3} - \frac{2e^3}{9} = \frac{6e^3}{9} - \frac{2e^3}{9} = \boxed{\frac{4e^3}{9}}$ Converges

Match

(*) $\lim_{t\to 0^+} t^3\cdot\ln(t^2)$ [$0\cdot(-\infty)$] $= \lim_{t\to 0^+}\frac{\ln(t^2)}{\frac{1}{t^3}}$ [$\frac{-\infty}{\infty}$; $\frac{1}{t^3} = t^{-3}$] $\overset{L'H}{=} \lim_{t\to 0^+}\frac{\frac{1}{t^2}\cdot(2t)}{\frac{-3}{t^4}}$ [$-\frac{t^4}{3}$] $= \lim_{t\to 0^+}\frac{-2t^4}{3t}$ (cancel) $= \lim_{t\to 0^+} -\frac{2t^3}{3} = 0$

Key Note: ln 0 is undefined, so must "sneak attack" 0 using Limit 0^+

1(b) $\int_e^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_e^t (\ln x)\cdot x^{-2}\,dx$ [IBP] $= \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_e^t + \int_e^t x^{-2}\,dx$

$u=\ln x$	$dv = x^{-2}\,dx$
$du = \frac{1}{x}\,dx$	$v = \frac{x^{-1}}{-1} = -\frac{1}{x}$

$= \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_e^t - \frac{1}{x}\Big|_e^t$

$= \lim_{t\to\infty} -\frac{\ln t}{t} + \frac{\ln e}{e} - \frac{1}{t} + \frac{1}{e}$ [$\frac{\infty}{\infty}$; $\ln e = 1$; $\frac{1}{t}\to 0$]

$\overset{L'H}{=} \lim_{t\to\infty} -\frac{\frac{1}{t}}{1} + \frac{1}{e} + \frac{1}{e} = \boxed{\frac{2}{e}}$ Converges

Match!

1(c) $\int_{-\infty}^{-3} \frac{8-x}{x^2+2x+5}\,dx = \lim_{t\to-\infty} \int_t^{-3} \frac{8-x}{x^2+2x+5}\,dx = \lim_{t\to-\infty} \int_t^{-3} \frac{8-x}{(x+1)^2+4}\,dx$

Complete Square

$(x+1)^2 = x^2+2x+1$
$+4 \qquad +4$

$u = x+1 \Rightarrow x = u-1$
$du = dx$

$x = t \Rightarrow u = t+1$
$x = -3 \Rightarrow u = -3+1 = -2$

$= \lim_{t\to-\infty} \int_{t+1}^{-2} \frac{8-(u-1)}{u^2+4}\,du = \lim_{t\to-\infty} \int_{t+1}^{-2} \frac{9-u}{u^2+4}\,du$

$= \lim_{t\to-\infty} \int_{t+1}^{-2} \frac{9}{u^2+4} - \frac{u}{u^2+4}\,du$

$= \lim_{t\to-\infty} \frac{9}{2}\arctan\left(\frac{u}{2}\right) - \frac{1}{2}\ln|u^2+4| \Big|_{t+1}^{-2}$

$= \lim_{t\to-\infty} \frac{9}{2}\arctan\left(\frac{-2}{2}\right) - \frac{1}{2}\ln 8 - \left(\frac{9}{2}\arctan\left(\frac{t+1}{2}\right) - \frac{1}{2}\ln|(t+1)^2+4|\right)$

($-\frac{\pi}{4}$, -1: Finite; Finite; $-\frac{\pi}{2}$, $-\infty$: Finite; ∞, ∞, $-\infty$)

$= \infty$ Diverges

Matches!

Finish all pieces
Find all arctan values.

1(d) $\int_{-4}^{-3} \frac{8-x}{x^2+2x-8}\,dx = \int_{-4}^{-3} \frac{8-x}{(x-2)(x+4)}\,dx = \lim_{t\to-4^+} \int_t^{-3} \frac{8-x}{(x-2)(x+4)}\,dx$

PFD

$(x-2)(x+4)\left(\frac{8-x}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4}\right)(x-2)(x+4)$

$8-x = A(x+4)+B(x-2)$
$= Ax+4A+Bx-2B$
$= (A+B)x+(4A-2B)$

Conditions

- $A+B=-1 \Rightarrow A=-1-B$
- $4A-2B=8$ $\quad 4(-1-B)-2B=8$
  $-4-4B-2B=8$
  $-6B=12$
  $B=-2$
  $A=-1-(-2)=1$

$= \lim_{t\to-4^+} \int_t^{-3} \frac{1}{x-2} - \frac{2}{x+4}\,dx$

$= \lim_{t\to-4^+} \ln|x-2| - 2\ln|x+4| \Big|_t^{-3}$

Justify size!

$= \lim_{t\to-4^+} \ln|-5| - 2\ln|-1| - \left(\ln|t-2| - 2\ln|t+4|\right)$

($\ln 5$: Finite; 0; $\ln 6$, -4: Finite; $-\infty$, -4^+, 0^+: blows up)

$= -\infty$ Diverges

Match!

Finish: Make sure to simplify $\ln 1 = 0$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+7} \hookrightarrow \int_1^{\infty} \frac{1}{x^2+4x+7}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4x+7}\,dx$

Complete Square

$u = x+2$
$du = dx$

$x=1 \Rightarrow u=3$
$x=t \Rightarrow u=t+2$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{(x+2)^2+3}\,dx$$

$$= \lim_{t\to\infty} \int_3^{t+2} \frac{1}{u^2+3}\,du$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Big|_3^{t+2}$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \left(\arctan\left(\frac{t+2}{\sqrt{3}}\right) - \arctan\left(\frac{3}{\sqrt{3}}\right)\right)$$

($\frac{t+2}{\sqrt{3}} \to \infty$, $\arctan \to \frac{\pi}{2}$; $\frac{3}{\sqrt{3}} \to \sqrt{3}$, $\arctan \to \frac{\pi}{3}$)

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}$$

Integral Converges

$\Rightarrow$ **Series Converges by Integral Test**

3(a) $\sum_{n=1}^{\infty} \frac{n^5+8}{8n^5+1}$ **Diverges by nTDT** because

$$\lim_{n\to\infty} \frac{n^5+8}{8n^5+1} \cdot \frac{\frac{1}{n^5}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{1+\frac{8}{n^5}}{8+\frac{1}{n^5}} = \frac{1}{8} \neq 0$$

($\frac{8}{n^5} \to 0$, $\frac{1}{n^5} \to 0$)

3(b) $\sum_{n=1}^{\infty} \frac{(n+5)^8}{\ln(n+5)}$ **Diverges by nTDT** because

$$\lim_{n\to\infty} \frac{(n+5)^8}{\ln(n+5)} \; \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{(x+5)^8}{\ln(x+5)} \; \frac{\infty}{\infty} \overset{L'H}{=} \lim_{x\to\infty} \frac{8(x+5)^7}{\frac{1}{x+5}} = \lim_{x\to\infty} 8(x+5)^8 = \infty \neq 0$$

($x+5$; $(x+5)^8 \to \infty$)

3(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^8} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{n^8}$ Converges p-Series $p=8>1$

$\Downarrow$ Original Series **Converges by the Absolute Convergence Test** (ACT)

OR

1. Pick $b_n = \frac{1}{n^8} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^8} = 0$ ($n^8 \to \infty$)
3. Terms Decreasing

   $b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_n$

Original Series Converges by the Alternating Series Test

3 (d) $\sum_{n=1}^{\infty} \frac{\ln 5}{(n+5)^8} + \frac{(-1)^n 8}{5^{2n+1}}$ split $= \sum_{n=1}^{\infty} \frac{\ln 5}{(n+5)^8} + \sum_{n=1}^{\infty} \frac{(-1)^n 8}{5^{2n+1}}$

$\ln 5 \sum_{n=1}^{\infty} \frac{1}{(n+5)^8}$ ☆ $+$ $-\frac{8}{5^3} + \frac{8}{5^5} - \frac{8}{5^7} + \dots$

Converges by CT

$r = -\frac{1}{5^2}$

Constant Multiple of Convergent Series is Convergent

Converges by Geometric Series Test with $|r| = \left|-\frac{1}{5^2}\right| = \frac{1}{25} < 1$

☆ $\sum_{n=1}^{\infty} \frac{1}{(n+5)^8} \approx \sum_{n=1}^{\infty} \frac{1}{n^8}$ Convergent p-Series $p = 8 > 1$

Bound Terms

$\frac{1}{(n+5)^8} \leq \frac{1}{n^8}$

↳ $\sum_{n=1}^{\infty} \frac{1}{(n+5)^8}$ Converges by CT → OR LCT

Original Series **Converges** because the Sum of Two Convergent Series is Convergent

3(e) $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \dots = -\sum_{n=1}^{\infty} \frac{1}{n}$ Note: not Alternating here

Constant Multiple of the Divergent Harmonic p-Series with $p=1$ is Divergent

Create a Series

4(a) Diverges by nTDT using L'H Rule

Examples: $\sum_{n=2}^{\infty} \frac{n^2}{\ln n}$ $\sum_{n=2}^{\infty} \frac{e^n}{\ln n}$ $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$ $\sum_{n=1}^{\infty} \left(1 - \frac{5}{n^2}\right)^{n^2}$

Continue on to prove your case.

Switch to Related function in x for L'H

Create a Series

4(b) Converges by Comparison Test. Need "Smaller than Converge" Comparison

Examples: $\sum_{n=1}^{\infty} \frac{n^2}{n^9+5}$ $\sum_{n=1}^{\infty} \frac{1}{9^n+5}$ $\sum_{n=1}^{\infty} \frac{3^n}{7^n+6}$ $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^5+4}$ $\sum_{n=1}^{\infty} \frac{\cos^2 n}{8^n+1}$

Sample. $\sum_{n=1}^{\infty} \frac{1}{n^6+9} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$ Converges p-Series $p = 6 > 1$

Bound Terms

$\frac{1}{n^6+9} \leq \frac{1}{n^6}$

$\Rightarrow$ Original Series also **Converges by Comparison Test**

Create a Series

4(c) Absolutely Convergent by Ratio Test

Examples. $\sum_{n=1}^{\infty} \frac{1}{n!}$ $\quad \sum_{n=1}^{\infty} \frac{e^n}{n!}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^n}$ $\quad \sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^n (2n)!}$ $\quad \sum_{n=1}^{\infty} \frac{n^6 n^n (n!)^2}{(3n)!}$

(Simplest?!)

Sample: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{1}{e} < 1$$

$(n+1)! = (n+1)\cdot n!$

$(n+1)^{n+1} = (n+1)^n \cdot (n+1)$

$\frac{n^n}{(n+1)^n} \to \frac{1}{e}$

$\Rightarrow$ **Series Converges Absolutely by Ratio Test**

Create a Series

4(d) Alternating Series Convergent by ACT

Examples: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6+1}$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^8+7}$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n n^6}{n^9+5}$

Sample:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5+n^4+n^3+n^2+n+1} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{n^5+n^4+n^3+n^2+n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n^5}$$

Converges p-Series $p=5>1$

Bound Terms

$$\frac{1}{n^5+n^4+n^3+n^2+n+1} \le \frac{1}{n^5}$$

Note: Limit Comparison also Works here

$\Rightarrow$ Absolute Series also Converges by the Comparison Test

ACT $\Leftarrow$

Original Series Converges by the Absolute Convergence Test

5(a) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^5+5n+8}{n^8+5}\right)$ A.S. $\rightarrow \sum_{n=1}^{\infty} \frac{n^5+5n+8}{n^8+5} \approx \sum_{n=1}^{\infty} \frac{n^5}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges p-Series $p=3>1$

$$\lim_{n\to\infty} \frac{\frac{n^5+5n+8}{n^8+5}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^8+5n^4+8n^3}{n^8+5} \cdot \frac{\frac{1}{n^8}}{\frac{1}{n^8}} = \lim_{n\to\infty} \frac{1+\frac{5}{n^4}+\frac{8}{n^5}}{1+\frac{5}{n^8}} = 1$$

Finite Non-Zero

$\Rightarrow$ Absolute Series also Converges by the Limit Comparison Test

$\Rightarrow$ Original Series is Absolutely Convergent by Definition

note: no need to test original series

5(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^5 \cdot n^n \cdot n!}{(2n+1)!}$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)^5(n+1)^{n+1}(n+1)!}{(2(n+1)+1)!}}{\frac{(-1)^n n^5 n^n n!}{(2n+1)!}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \cdot \frac{(n+1)^{n+1}}{n^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!}$$

$(n+1)^{n+1} = (n+1)^n(n+1)$, $(n+1)! = (n+1)n!$, $(2n+3)! = (2n+3)(2n+2)(2n+1)!$

Partner these term pieces

$\frac{(n+1)^5}{n^5} = \left(\frac{n+1}{n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right)^5 = \left(\frac{1+\frac{1}{n}}{1}\right)^5 \to 1$ or $\left(1+\frac{1}{n}\right)^5 \to 1$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \left(\frac{n+1}{2n+3}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{n+1}{2n+2}\right)$$

$2n+2 = 2(n+1)$ — Don't Drop

$\frac{(n+1)^n}{n^n} \to e$

$$= \lim_{n\to\infty} \frac{e}{2}\left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right) = \frac{e}{4} < 1$$

$\Rightarrow$ the Series Converges Absolutely by the Ratio Test

5(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+8}$ A.S. → $\sum_{n=1}^{\infty} \frac{1}{5n+8} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges p-Series p=1

$\lim_{n\to\infty} \frac{\frac{1}{5n+8}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+8} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{5+\frac{8}{n}} = \frac{1}{5}$ Finite Non-Zero

⇒ Absolute Series also Diverges by the Limit Comparison Test

AST

1. Pick $b_n = \frac{1}{5n+8} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{5n+8} = 0$

3. Terms Decreasing

$b_{n+1} = \frac{1}{5(n+1)+8} \leq \frac{1}{5n+8} = b_n$

$5n+13$

Original Series Converges by the Alternating Series Test

OR show Related Function

$f(x) = \frac{1}{5x+8}$ has $f'(x) = \frac{-5}{(5x+8)^2} < 0$

Original Series is Conditionally Convergent by Definition