Note: This Answer key may contain some short-hand notations, using A.S for the Absolute Series and O.S. for the Original Series, and maybe some shorthand labels for each of the Convergence Tests.

Determine whether the given series is **Absolutely Convergent**, **Conditionally Convergent**, or **Divergent**. Name any Convergence Test(s) you use, and justify all of your work.

1. $\sum_{n=1}^{\infty} (-1)^n \frac{n^4+7}{n^7+4}$

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^4+7}{n^7+4} \quad \xrightarrow[\text{A.S.}]{1^{st}} \quad \sum_{n=1}^{\infty} \frac{n^4+7}{n^7+4} \approx \sum_{n=1}^{\infty} \frac{n^4}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Conv. p-Series
$p = 3 > 1$

$$\lim_{n\to\infty} \frac{\frac{n^4+7}{n^7+4}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^7+7n^3}{n^7+4} \cdot \frac{\left(\frac{1}{n^7}\right)}{\left(\frac{1}{n^7}\right)}$$

$$= \lim_{n\to\infty} \frac{1+\cancelto{0}{\frac{7}{n^4}}}{1+\cancelto{0}{\frac{4}{n^7}}} = 1 \text{ Finite and Non-zero}$$

no need to analyze O.S.

$\boxed{\text{A.S. Converges}}$ by Limit Comparison Test

$\boxed{\text{Original Series Absolutely Convergent } \textit{by Definition}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ 3^n\ n!\ n^n}{n^3\ (2n)!}$

Try Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}\ 3^{n+1}\ (n+1)!\ (n+1)^{n+1}}{(n+1)^3\ (2(n+1))!}}{\dfrac{\cancel{(-1)^n}\ 3^n\ n!\ n^n}{n^3\ (2n)!}} \right|$$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n^3}{(n+1)^3} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{3^n}\cdot 3}{\cancel{3^n}} \cdot \frac{(n+1)\cancel{n!}}{\cancel{n!}} \cdot \frac{(n+1)^n(n+1)}{n^n} \cdot \left(\frac{n}{n+1} \frac{\frac{1}{n}}{\frac{1}{n}} \right)^3 \cdot \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} (3) \cdot \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \cdot \cancel{\frac{(n+1)^n}{n^n}}^{\to e} \cdot \cancel{\left(\frac{1}{1+\frac{1}{n}} \right)^3}^{\to 1}$$

$$= \lim_{n\to\infty} (3e) \left(\frac{n+1}{2n+2} \right) \left(\frac{n+1}{2n+1} \right)$$

$$= \lim_{n\to\infty} (3e) \left(\frac{\cancel{n+1}}{2\,\cancel{(n+1)}} \right) \left(\frac{n+1}{2n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \right)$$

$$= \lim_{n\to\infty} \frac{3e}{2} \left(\frac{1+\cancel{\frac{1}{n}}^{\to 0}}{2+\cancel{\frac{1}{n}}^{\to 0}} \right) = \frac{3e}{4} > 1$$

Therefore, the Original Series $\boxed{\text{Diverges by the Ratio Test}}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+7}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+7} \quad \xhookrightarrow[\text{A.S.}]{1^{st}} \quad \sum_{n=1}^{\infty} \frac{1}{4n+7} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$

Diverges. p-Series
Harmonic $p=1$

$\swarrow 2^{nd}$

1. $b_n = \frac{1}{4n+7} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{4n+7} = \frac{1}{\infty} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{4n+11} \leq \frac{1}{4n+7} = b_n$$

$$\lim_{n\to\infty} \frac{\frac{1}{4n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{4n+7} \cdot \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{4+\frac{7}{n}} = \frac{1}{4}$$ Finite and Non-zero

$\boxed{\text{O.S Converges}}$ by AST

$\boxed{\text{A.S Diverges}}$ by LCT

$\boxed{\begin{array}{c}\text{Original Series}\\ \text{Conditionally}\\ \text{Convergent}\\ \textit{by Definition}\end{array}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (2n)!\ \ln n}{5^n\ (n!)^2}$

Try Ratio Test:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}\ (2(n+1))!\ \ln(n+1)}{5^{n+1}((n+1)!)^2}}{\dfrac{\cancel{(-1)^n}\ (2n)!\ \ln n}{5^n\ (n!)^2}} \right| \\
&= \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \frac{\ln(n+1)}{\ln n} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{(n!)^2}{((n+1)!)^2} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}} \cdot \cancelto{1}{\frac{\ln(n+1)}{\ln n}}{}^{*} \cdot \frac{\cancel{5^n}}{\cancel{5^n}\cdot 5} \cdot \frac{\cancel{(n!)^2}}{(n+1)^2\cancel{(n!)^2}} \\
&= \lim_{n\to\infty} \frac{1}{5} \cdot \frac{2\,\cancel{(n+1)}}{\cancel{n+1}} \cdot \left(\frac{2n+1}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \right) \\
&= \lim_{n\to\infty} \frac{2}{5} \cdot \left(\frac{2+\cancelto{0}{\frac{1}{n}}}{1+\cancelto{0}{\frac{1}{n}}} \right) = \frac{4}{5} < 1
\end{aligned}
$$

Therefore, the Original Series $\boxed{\text{Converges Absolutely by the Ratio Test}}$

*L'H Rule on the log piece

$$
\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n}^{\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x}^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1}^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{1} = 1
$$

5. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+5}{n^9+3}$

$$\sum_{n=1}^{\infty}(-1)^n \frac{n+5}{n^9+3} \quad \xhookrightarrow[\text{A.S.}]{1^{st}} \quad \sum_{n=1}^{\infty} \frac{n+5}{n^9+3} \approx \sum_{n=1}^{\infty} \frac{n}{n^9} = \sum_{n=1}^{\infty} \frac{1}{n^8}$$

Conv. p-Series
$p = 8 > 1$

$$\lim_{n\to\infty} \frac{\frac{n+5}{n^9+3}}{\frac{1}{n^8}} = \lim_{n\to\infty} \frac{n^9+5n^8}{n^9+3} \cdot \frac{\left(\frac{1}{n^9}\right)}{\left(\frac{1}{n^9}\right)}$$

$$= \lim_{n\to\infty} \frac{1+\cancelto{0}{\frac{5}{n}}}{1+\cancelto{0}{\frac{3}{n^9}}} = 1 \text{ Finite and Non-zero}$$

no need to analyze O.S.

$\boxed{\text{A.S. Converges}}$ by Limit Comparison Test

Original Series Absolutely Convergent *by Definition*

6. $\sum_{n=1}^{\infty} (-1)^n \dfrac{n}{n^2+6}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{n^2+6} \quad \xrightarrow[\text{A.S.}]{1^{st}} \quad \sum_{n=1}^{\infty} \frac{n}{n^2+6} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$

$\swarrow 2^{nd}$

Diverges. p-Series
Harmonic $p = 1$

1. $b_n = \dfrac{n}{n^2+6} > 0$

2. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{n}{n^2+6} \cdot \dfrac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$

$= \lim\limits_{n\to\infty} \dfrac{\cancelto{0}{\frac{1}{n}}}{1+\frac{6}{n^2}} = 0$

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+6}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+6} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{1+\cancelto{0}{\frac{6}{n^2}}} = \frac{1}{6} \text{ Finite/Non-zero}$$

3. Terms Decreasing
The related function $f(x) = \dfrac{x}{x^2+6}$
has derivative $f'(x) = \dfrac{-x^2+6}{(x^2+6)^2 \oplus} < 0$
for $x > \sqrt{6}$

$\boxed{\text{O.S Converges}}$ by AST

$\boxed{\text{A.S Diverges}}$ by LCT

$\boxed{\begin{array}{c}\text{Original Series} \\ \text{Conditionally} \\ \text{Convergent} \\ \textit{by Definition}\end{array}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^2\ 2^{4n}\ n^n}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\dfrac{(3(n+1))!}{((n+1)!)^2\ 2^{4(n+1)}(n+1)^{n+1}}}{\dfrac{(3n)!}{(n!)^2\ 2^{4n}\ n^n}}\right|$$

$$= \lim_{n\to\infty}\frac{(3n+3)!}{(3n)!}\cdot\frac{(n!)^2}{((n+1)!)^2}\cdot\frac{2^{4n}}{2^{4n+4}}\cdot\frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty}\frac{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}{\cancel{(3n)!}}\cdot\frac{\cancel{(n!)^2}}{(n+1)^2\cancel{(n!)^2}}\cdot\frac{\cancel{2^{4n}}}{\cancel{2^{4n}}\cdot 2^4}\cdot\frac{n^n}{(n+1)^n\cdot(n+1)}$$

$$= \lim_{n\to\infty}\frac{3\cancel{(n+1)}}{\cancel{n+1}}\cdot\left(\frac{3n+2}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)\cdot\left(\frac{3n+1}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)\cdot\frac{1}{16}\cdot\cancelto{\frac{1}{e}}{\frac{n^n}{(n+1)^n}}$$

$$= \lim_{n\to\infty}(3)\cdot\cancelto{3}{\left(\frac{3+\cancelto{0}{\frac{2}{n}}}{1+\cancelto{0}{\frac{1}{n}}}\right)}\cdot\cancelto{3}{\left(\frac{3+\cancelto{0}{\frac{1}{n}}}{1+\cancelto{0}{\frac{1}{n}}}\right)}\cdot\frac{1}{16}\cdot\cancelto{\frac{1}{e}}{\frac{n^n}{(n+1)^n}} = \frac{27}{16e} < 1$$

Therefore, the Original Series $\boxed{\text{Converges Absolutely by the Ratio Test}}$

8. Use **Two Different** methods to show that $\displaystyle\sum_{n=2}^{\infty}\frac{e^n}{\ln n}$ Diverges.

First, the series $\boxed{\text{Diverges by the } n^{th} \text{ term Divergence Test}}$ because

$$\lim_{n\to\infty}\frac{e^n}{\ln n}^{\frac{\infty}{\infty}} = \lim_{x\to\infty}\frac{e^x}{\ln x}^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{e^x}{\dfrac{1}{x}} = \lim_{x\to\infty} xe^x = \infty \neq 0$$

Second, the series $\boxed{\text{Diverges by the Ratio Test}}$ because

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{e^{n+1}}{\ln(n+1)}}{\dfrac{e^n}{\ln n}}\right| = \lim_{n\to\infty}\frac{e^{n+1}}{e^n}\cdot\cancelto{1}{\frac{\ln n}{\ln(n+1)}} \overset{*}{=} \lim_{n\to\infty}\frac{\cancel{e^n}\cdot e}{\cancel{e^n}}(1) = e > 1$$

*L'H Rule on the log piece

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}^{\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)}^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{x+1}{x}^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{1} = 1$$

Note: can also finish the L'H Limit with algebra

$$\ldots = \lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} 1 + \overset{0}{\cancel{\frac{1}{x}}} = 1$$

OR you can finish it with *divides-by-algebra*

$$\ldots = \lim_{x\to\infty} \frac{x+1}{x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1+\overset{0}{\cancel{\frac{1}{x}}}}{1} = 1$$

9. Use the **Absolute Convergence Test** to show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6+7}$ Converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6+7}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^6+7} \quad \xhookrightarrow[\text{A.S.}]{1^{st}} \quad \sum_{n=1}^{\infty} \frac{1}{n^6+7} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$$

Conv. p-Series
$p = 6 > 1$

Bound Terms: $\dfrac{1}{n^6+7} < \dfrac{1}{n^6}$

no need to analyze O.S.

$\boxed{\text{A.S. Converges}}$ by Comparison Test

$$\boxed{\begin{array}{c}\text{Original Series} \\ \text{Convergent} \\ \textit{by Absolute} \\ \textit{Convergence Test}\end{array}}$$

Check the statement of the Absolute Convergence Test: Given a series, if the Absolute series converges, then the Original Series Converges.

10. Use the Series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ to show that $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Use the Ratio Test on the Series first.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\cancel{n}!}{\cancel{n}!} \cdot \frac{n^n}{(n+1)^n \cancel{(n+1)}} = \lim_{n\to\infty} \cancelto{\frac{1}{e}}{\frac{n^n}{(n+1)^n}} = \frac{1}{e} < 1$$

Therefore, the Series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ Converges (Absolutely) by the Ratio Test.

As a result we can conclude that the TERMS of the series shoot to 0 as n explodes to infinity. That is, $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0$ because otherwise, by contradiction, IF the limit of the terms was not equal to 0, then the series would diverge by the n^{th} Term Divergence Test, which would contradict our Ratio Test convergence work here.