Math 121 Introduction to Sequences

The objective of this handout is to introduce the topic of Infinite Sequences. We will pay careful attention to *what* they are, *how* they are defined, *how* to decide convergence of each, as well as *how* they are related. We will also list a few classic examples of Sequences, and finally we will start to study the limiting values, if they exist.

First, this topic is a first stop in a larger plan. We will spend about 6 weeks studying Infinite Sums of Real numbers, called Infinite Series.

Really Long Term Goal: To Study the relationship between Series and Functions

Long Term Goal: To Study Infinite Series

Short Term Goal: To Study Infinite Sequences.

Throughout this study of sequences and series, we will pay particular attention to the use of *Conditional Statements* in the form **IF** ... **THEN** ...

Let's start with the definition of an Infinite Sequence.

• Infinite Sequences Definition

Definition: An Infinite **Sequence** of real numbers is an ordered, unending list of numbers.

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_{n-1},\ a_n,\ a_{n+1},\ \ldots$$

Here, a_n is acting as the n^{th} term of the infinite list where the positive integer counters $n = 1, 2, 3, \ldots$, are keeping the values organized in placement order in the infinite list.

a_1 represents the first term in the list

a_2 represents the second term in the list

$\vdots$

a_n represents the n^{th} term in the list

a_{n+1} represents the $(n+1)^{st}$ term in the list

$\vdots$

It is key to recognize that this list is infinite and goes on forever, so we need to always make sure to add the dot, dot, dot symbol ... at the end every time. Terms of an Infinite List are separated by commas.

It can be difficult to represent an infinite list of numbers, so we abbreviate the sequence terms with a concise notation. Here n is an integer counter for the terms.

$$\{a_n\}_{n=1}^{\infty} \qquad \text{or} \qquad \{a_n\}_1^{\infty} \qquad \text{or} \qquad \{a_n\}_{n\geq 1} \qquad \text{or even} \quad \{a_n\} \text{ for short}$$

To clarify, the bracket notation should instantly make you think of an infinite **LIST**.

$$\{a_n\}_{n=1}^{\infty} = a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_{n-1},\ a_n,\ a_{n+1},\ \ldots$$

Here, again, n is acting as the *Counter Index* $n = 1, 2, 3, \ldots$.

$n = 1$ counts the first term in the list

$n = 2$ counts the second term in the list

$\vdots$

n counts the n^{th} term in the list

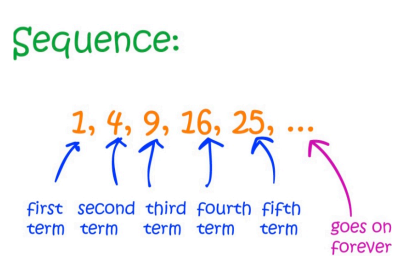
$n + 1$ counts the $(n + 1)^{st}$ term in the list

$\vdots$

Notation is key here, but we also have flexibility. We can use a different variable to represent the counter index $n = 1, 2, 3, \ldots$. That is,

$$\{a_n\}_{n=1}^{\infty} = \{a_i\}_{i=1}^{\infty} = \{a_j\}_{j=1}^{\infty} = \{a_k\}_{k=1}^{\infty} = a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots$$

Because we typically use n to represent the postive counting integers, $n = 1, 2, 3\ldots$, for now, we agree that the counter index starts at $n = 1$. When we study series in the future, that may change.



• **Infinite Sequences Examples** Sequences come in many forms, including increasing, decreasing, or even alternating. Here is an example of a decreasing sequence.

Ex: $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5}, \ldots$

Here the sequence term formula is given by $a_n = \frac{1}{n}$, as posted between the brackets { and }.

To clarify the partnership between the counter numbers $n = 1, 2, 3\ldots$ and the output of the valued terms of each sequence determined by a given n^{th} term formula, study this ordering

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \underset{\text{1st term}}{\overset{n=1}{1}},\ \underset{\text{2nd term}}{\overset{n=2}{\frac{1}{2}}},\ \underset{\text{3rd term}}{\overset{n=3}{\frac{1}{3}}},\ \underset{\text{4th term}}{\overset{n=4}{\frac{1}{4}}},\ \underset{\text{5th term}}{\overset{n=5}{\frac{1}{5}}},\ \ldots$$

Chart of Examples

SEQUENCE $\{a_n\}_{n=1}^{\infty}$	n^{th} term a_n	LIST $a_1, a_2, a_3, a_4, \ldots$	LIST Appears to Approach as $n \longrightarrow \infty$
$\{n\}_{n=1}^{\infty}$	$a_n = n$	$1, 2, 3, 4, 5, 6, \ldots$	$\longrightarrow \infty$??
$\{6\}_{n=1}^{\infty}$	$a_n = 6$	$6, 6, 6, 6, 6, 6, \ldots$	$\longrightarrow 6$??
$\{2^n\}_{n=1}^{\infty}$	$a_n = 2^n$	$2, 4, 8, 16, 32\ldots$	$\longrightarrow \infty$??
$\{n^2\}_{n=1}^{\infty}$	$a_n = n^2$	$1, 4, 9, 16, 25, \ldots$	$\longrightarrow \infty$??
$\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$	$a_n = \frac{1}{n^2}$	$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$	$\longrightarrow 0$??
$\left\{\frac{1}{(10)^n}\right\}_{n=1}^{\infty}$	$a_n = \frac{1}{(10)^n}$	$\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10,000}, \ldots$	$\longrightarrow 0$??
$\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$	$a_n = \frac{(-1)^n}{n}$	$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \ldots$	$\longrightarrow 0$??

QUESTION: Do these Infinite Sequences have Limiting values, as the number of terms $n \to \infty$?

$$\boxed{\text{The Sequence } \{a_n\}_{n=1}^{\infty} \textbf{ Converges} \text{ to the Limit value } L \text{ if } \lim_{n\to\infty} a_n = L \text{ for some } \textit{Finite} \text{ Limit } L.}$$

Otherwise, we say the Sequence **Diverges**. A sequence can diverge when terms explode in size towards either $+\infty$ or $-\infty$, as the counter n grows. OR the terms can diverge by oscillation.

Many questions arise:

- Does a list of numbers approach a fixed number or Limit L?

- Not only does the list get close to a limit L, but does it remain close to L?

- If some limit L exists, how do we compute that number L?

- **Limit Computations:** There are several methods for **computing** Limits of sequences. We will just mention a few here to get started. These computations can range from lighter natural instincts of *size* all the way to a very complex proof argument at the higher level. Many involve foundational growth and algebra arguments.

Plan of attack: Given an Infinite Sequence $\{a_n\}_{n=1}^{\infty}$, step aside from the bracket notation and study the Limiting value of (just) the term formula a_n, letting n grow large towards ∞. That is,

$$\boxed{\text{Given } \{a_n\}_{n=1}^{\infty} \quad \text{study } \lim_{n\to\infty} a_n}$$

To study the Limit of a sequence, make sure to drop the notational brackets in the Limit. No need to write out the first few terms. Our instincts will prove correct in many cases:

Ex: Given $\{n^2\}_{n=1}^{\infty}$ then $\lim\limits_{n\to\infty} n^2 = \lim\limits_{n\to\infty} \cancelto{\infty}{n^2} = \infty$. The Sequence Diverges.

Ex: Given constant sequence $\{6\}_{n=1}^{\infty}$, then $\lim\limits_{n\to\infty} 6 = 6$. The Sequence Converges to the finite value 6.

Ex: Given $\left\{\dfrac{1}{n^3}\right\}_{n=1}^{\infty}$ then $\lim\limits_{n\to\infty} \dfrac{1}{n^3} = \lim\limits_{n\to\infty} \cancelto{0}{\dfrac{1}{\cancelto{\infty}{n^3}}} = 0$. The sequence Converges to 0.

Note that the Limit Laws still work in this setting. Our Calculus instincts/permissions with Limits are still in place, even though our measure of change is with respect to $n \to \infty$ where n represents positive counting integers rather than all real numbers. We will see they are related in the future. For now, here are a few of the Limit Laws for sequences.

- **Limit Laws for Sequences**

If $\{a_n\}$ and $\{b_n\}$ are both convergent sequences and c is a constant, then

- $\lim\limits_{n\to\infty} a_n \pm b_n = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n$
- $\lim\limits_{n\to\infty} c = c$
- $\lim\limits_{n\to\infty} a_n \cdot b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ provided the denominator limit is non-zero

- **Limit Examples:** Question: Determine whether the given sequence Converges or Diverges. If it converges, find the Limit.

- **Limits of Stacks of Polynomials:** These should remind you of Limits at Infinity (like Horizontal Asymptotes) from Calculus I.

Ex: Consider $\left\{\dfrac{5n^3-6n^2+9}{7n^3+4n-2}\right\}_{n=1}^{\infty}$. Study the limiting value of the terms.

$$\lim_{n\to\infty}\frac{5n^3-6n^2+9}{7n^3+4n-2} = \lim_{n\to\infty}\frac{5n^3-6n^2+9}{7n^3+4n-2}\cdot\frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3}\right)} = \lim_{n\to\infty}\frac{5-\overset{0}{\cancel{\frac{6}{n}}}+\overset{0}{\cancel{\frac{9}{n^3}}}}{7+\overset{0}{\cancel{\frac{4}{n^2}}}-\overset{0}{\cancel{\frac{2}{n^3}}}} = \boxed{\frac{5}{7}} \quad \text{Finite}$$

We say the Sequence Converges to $\dfrac{5}{7}$ because the terms settle to that value for large n.

Ex: Consider $\left\{\dfrac{7n^2-3n-2}{9-8n^2}\right\}_{n=1}^{\infty}$. Study the limiting value of the terms.

$$\lim_{n\to\infty}\frac{7n^2-3n-2}{9-8n^2} = \lim_{n\to\infty}\frac{7n^2-3n-2}{9-8n^2}\cdot\frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty}\frac{7-\overset{0}{\cancel{\frac{3}{n}}}+\overset{0}{\cancel{\frac{2}{n^2}}}}{\overset{0}{\cancel{\frac{9}{n^2}}}-8} = \boxed{-\frac{7}{8}} \quad \text{Finite}$$

We say the Sequence Converges to $-\dfrac{7}{8}$ because the terms settle to that value for large n.

- **Factorials, Definition**

Dealing with large products of numbers can be complicated. Shorthand notation is helpful.

Definition: $\boxed{n \text{ Factorial}}$ is the product of all positive integers less than or equal to the number n. This large product is denoted with the number n and an exclamation point.

$$\boxed{n! \overset{\text{definition}}{=} n(n-1)(n-2)(n-3)\cdot \ldots \cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

Study a few examples:

Ex: $3! = 3\cdot 2\cdot 1 = 6$

Ex: $6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$

Ex: $(12)! = 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 470,001,600$

Ex: Note that $0!$ is defined to be equal to 1.

These Factorial values $n!$ grow very large, very quickly, as $n \to \infty$. That is, $\lim\limits_{n\to\infty} \cancelto{\infty}{n}! = \infty$

Note that n Factorial can be written several ways

$$\begin{aligned} n! \quad &= n\,\underbrace{(n-1)\,\overbrace{(n-2)(n-3)\cdot \ldots \cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}^{(n-2)!}}_{(n-1)!} \\ &\overset{\text{or}}{=} n\cdot (n-1)! \\ &\overset{\text{or}}{=} n\cdot (n-1)\cdot (n-2)! \end{aligned}$$

For example, we can write

$$5! = 5\cdot \underbrace{4\cdot \overbrace{3\cdot 2\cdot 1}^{3!}}_{4!} \overset{\text{or}}{=} 5\cdot 4! \overset{\text{or}}{=} 5\cdot 4\cdot 3!$$

- **Factorial Algebra**

Since Factorials are large products, we can often cancel values when they are stacked against other Fatorials.

Ex: Simplify $\quad \dfrac{6!}{4!} = \dfrac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1} = \dfrac{6\cdot 5\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}{\cancel{4\cdot 3\cdot 2\cdot 1}} = 6\cdot 5 = 30$

Ex: Simplify $\quad \dfrac{5!}{8!} = \dfrac{5\cdot 4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \dfrac{\cancel{5\cdot 4\cdot 3\cdot 2\cdot 1}}{8\cdot 7\cdot 6\cdot \cancel{5\cdot 4\cdot 3\cdot 2\cdot 1}} = \dfrac{1}{8\cdot 7\cdot 6} = \dfrac{1}{336}$

• **Factorial Limits** We can now study Limiting values for Sequences involving Factorials.

Q: Determine whether the sequence Converges or Diverges. If it converges, find the Limit.

Ex: $\left\{ \dfrac{n!}{(n+1)!} \right\}_{n=1}^{\infty}$ Study the limiting value of the terms $a_n = \dfrac{n!}{(n+1)!}$.

TIP: Remember to drop the bracket notation.

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n(n-1)(n-2)(n-3)\cdot\ldots\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(n+1)n(n-1)(n-2)(n-3)\cdot\ldots\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n(n-1)(n-2)(n-3)\cdot\ldots\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{\text{(n+1)}\,\cancel{n(n-1)(n-2)(n-3)\cdot\ldots\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} \frac{1}{\underbrace{n+1}_{\infty}} = 0$$

OR try this again using shorthand notation

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)\cdot n!} = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cdot \cancel{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} \frac{1}{\underbrace{n+1}_{\infty}} = 0$$

We say this Sequence Converges to 0.

Ex: $\left\{ \dfrac{(n+1)!}{(n-1)!} \right\}_{n=1}^{\infty}$ Study

$$\lim_{n\to\infty} \frac{(n+1)!}{(n-1)!} = \lim_{n\to\infty} \frac{(n+1)\cdot n\cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = \lim_{n\to\infty} \underbrace{(n+1)\cdot n}_{\infty} = \infty$$

We say this Sequence Diverges to ∞.

Ex: $\left\{ \dfrac{(2n)!}{(2n+2)!} \right\}_{n=1}^{\infty}$ Study

$$\lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} = \lim_{n\to\infty} \frac{1}{\underbrace{(2n+2)\cdot(2n+1)}_{\infty}} = 0$$

We say this Sequence Converges to 0.

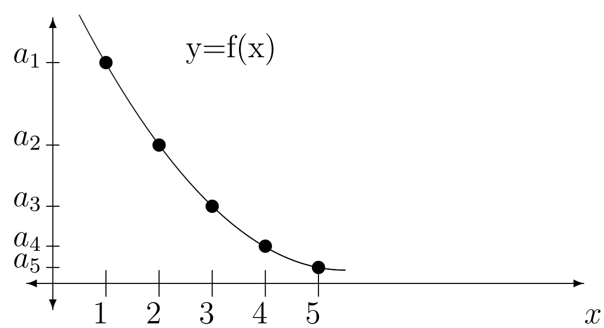
Next, we will study a careful approach to computing complicated Limits. To come in class...

• **Function Relationship** Some Limits involve using L'Hôpital's Rule. We will need to work with *related functions* because we may need to take derivatives.

Think of the terms of a sequence as defined by a single, related function where the terms $a_n = f(n)$.

For instance, the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ has a related function $f(x) = \frac{1}{x}$

Essentially, we have a formula for the terms of the sequence. The terms of the sequence can be represented as output plot points on the graph of $f(x)$, when you restrict the domain of f to a domain of positive integers n. Below $f(x)$ is drawn as the curve, and the dots are the sequence terms plotted along this curve. Since the sequence terms lie along the curve $f(x)$, it makes sense that the behavior of the sequence terms might be related to the behavior of the function.



Theorem: Given a sequence $\{a_n\}$, if there is related function $f(x)$ so that the terms of our sequence $a_n = f(n)$ and if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$

Here we can use all of our previous limit techniques for functions. Please note that if you are going to apply L'H Rule, then technically you must really step aside and look at the related function f of x, since L'H Rule is not stated for terms of sequences.

Example: Does the sequence $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ converge?

$$\lim_{n\to\infty} \underbrace{\frac{\ln n}{n}}_{Dots} = \lim_{x\to\infty} \underbrace{\frac{\ln x}{x}}_{Function} \overset{\left(\frac{\infty}{\infty}\right)}{} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Finally, our given sequence Converges to the finite value 0.

Key Point: It is not true that the terms $\frac{\ln n}{n}$ (where $n = 1, 2, 3, \ldots$) are equal to the function $\frac{\ln x}{x}$, but they share the same *Limit*, if it exists.