

Exam #1 Answer Key Fall 2021

$$1(a) \lim_{x \rightarrow 0} \frac{\ln(1-5x) + \arcsin(5x)}{3xe^x - \arctan(3x)} = \lim_{x \rightarrow 0} \frac{\frac{-5}{1-5x} + \frac{5}{\sqrt{1-(5x)^2}}}{\frac{3xe^x + 3e^x - 3}{1+(3x)^2}}$$

prep $-5(1-5x)^{-1}$ prep $5(1-25x^2)^{-1/2}$

$$= \lim_{x \rightarrow 0} \frac{5(1-5x)^{-2}(-5) - \frac{5}{2}(1-25x^2)^{-3/2}(-50x)}{3xe^x + 3e^x + 3e^x + 3(1+9x^2)^{-2}(18x)} = \lim_{x \rightarrow 0} \frac{\frac{-25}{(1-5x)^2} + \frac{125x}{(1-25x^2)^{3/2}}}{3xe^x + 6e^x + \frac{54x}{(1+9x^2)^2}} = \frac{-25}{6}$$

Match!

$$(b) \lim_{x \rightarrow 0^+} x^5 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^5}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x} \cdot \left(\frac{-x^6}{5}\right)}{\frac{-5}{x^6}} = \lim_{x \rightarrow 0^+} \frac{-x^5}{5} = 0$$

Match!

$$(c) \lim_{x \rightarrow \infty} \left(1 - \arcsin\left(\frac{2}{x^4}\right)\right)^{x^4} = e^{\lim_{x \rightarrow \infty} x^4 \ln\left(1 - \arcsin\left(\frac{2}{x^4}\right)\right)}$$

Key: Show All Steps here

$$= e^{\lim_{x \rightarrow \infty} \frac{\ln\left(1 - \arcsin\left(\frac{2}{x^4}\right)\right)}{\frac{1}{x^4}}}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{1}{1 - \arcsin\left(\frac{2}{x^4}\right)} \cdot \left(\frac{-1}{\sqrt{1 - \left(\frac{2}{x^4}\right)^2}}\right) \cdot \frac{-8}{x^5}}$$

what cancels?

$$= e^{-2}$$

Match

$$2. \int_e^{e^3} \frac{1}{x[3+(\ln x)^2]} dx = \int_1^3 \frac{1}{3+u^2} du = \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) \Big|_1^3$$

$$\begin{aligned} u &= \ln x \\ du &= \frac{1}{x} dx \end{aligned}$$

$$\begin{aligned} x &= e \Rightarrow u = \ln e = 1 \\ x &= e^3 \Rightarrow u = \ln e^3 = 3 \end{aligned}$$

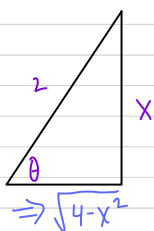
$$= \frac{1}{\sqrt{3}} \left(\arctan\left(\frac{3}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right) \right)$$

$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{1}{\sqrt{3}} \left(\frac{\pi}{6} \right) = \frac{\pi}{6\sqrt{3}}$$

$$3. \int \frac{1}{(4-x^2)^{3/2}} dx = \int \frac{1}{(\sqrt{4-x^2})^3} dx = \int \frac{1}{(\sqrt{4-4\sin^2\theta})^3} \cdot 2\cos\theta d\theta$$

Trig. Sub

$$\begin{aligned} x &= 2\sin\theta \rightarrow \sin\theta = \frac{x}{2} \\ dx &= 2\cos\theta d\theta \end{aligned}$$



$$= \int \frac{1}{2^3 \cos^3\theta} \cdot 2\cos\theta d\theta = \frac{1}{2^2} \int \frac{1}{\cos^2\theta} d\theta$$

$$= \frac{1}{4} \int \sec^2\theta d\theta = \frac{1}{4} \tan\theta + C = \frac{1}{4} \left(\frac{x}{\sqrt{4-x^2}} \right) + C$$

$$4. \int x^2 \arcsin x dx = \frac{x^3}{3} \arcsin x - \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx$$

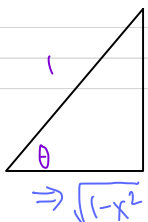
IBP

$$\begin{aligned} u &= \arcsin x & dv &= x^2 dx \\ du &= \frac{1}{\sqrt{1-x^2}} dx & v &= \frac{x^3}{3} \end{aligned}$$

$$= \frac{x^3}{3} \arcsin x - \frac{1}{3} \int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta d\theta$$

Trig. Sub

$$\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta d\theta \end{aligned}$$



$$\Rightarrow \cos\theta = \sqrt{1-x^2}$$

$$= \frac{x^3}{3} \arcsin x - \frac{1}{3} \int \sin^2\theta \cdot \sin\theta d\theta$$

$$= \frac{x^3}{3} \arcsin x - \frac{1}{3} \int \sin^2\theta \cdot \sin\theta d\theta$$

"Isolate"

u-sub

$u = \cos \theta$
$du = -\sin \theta d\theta$
$-du = \sin \theta d\theta$

$$= \frac{x^3}{3} \arcsin x - \frac{1}{3} \int (1 - \cos^2 \theta) \sin \theta d\theta$$

"Convert"

$$= \frac{x^3}{3} \arcsin x + \frac{1}{3} \int 1 - u^2 du$$

"Finish with u-sub"

$$= \frac{x^3}{3} \arcsin x + \frac{1}{3} \left(u - \frac{u^3}{3} \right) + C$$

undo u-sub

$$= \frac{x^3}{3} \arcsin x + \frac{1}{3} \left(\cos \theta - \frac{\cos^3 \theta}{3} \right) + C$$

undo Trig Sub

$$= \frac{x^3}{3} \arcsin x + \frac{1}{3} \left(\sqrt{1-x^2} - \frac{(\sqrt{1-x^2})^3}{3} \right) + C$$

\uparrow or $(1-x^2)^{3/2}$

$$5. \int_1^{e^2} \sqrt{x} \ln x dx = \frac{2}{3} x^{3/2} \ln x \Big|_1^{e^2} - \frac{2}{3} \int_1^{e^2} \frac{x^{3/2}}{x} dx$$

IBP

$u = \ln x$	$dv = \sqrt{x} dx$
$du = \frac{1}{x} dx$	$v = \frac{2}{3} x^{3/2}$

$$= \frac{2}{3} x^{3/2} \ln x \Big|_1^{e^2} - \frac{2}{3} \left(\frac{2}{3} x^{3/2} \right) \Big|_1^{e^2}$$

$$= \frac{2}{3} x^{3/2} \ln x \Big|_1^{e^2} - \frac{4}{9} x^{3/2} \Big|_1^{e^2}$$

$$= \frac{2}{3} (e^2)^{3/2} \ln(e^2) - \frac{2}{3} \ln 1 - \left(\frac{4}{9} (e^2)^{3/2} - \frac{4}{9} \right)$$

$$= \frac{2}{3} e^3 \cdot (2) - \frac{4}{9} e^3 + \frac{4}{9}$$

$$= \frac{4}{3} e^3 - \frac{4}{9} e^3 + \frac{4}{9} = \frac{8}{9} e^3 + \frac{4}{9} = \frac{8e^3 + 4}{9} \text{ Match!}$$

$$6. \int \frac{1}{(x^2+4)^2} dx = \int \frac{1}{(4\tan^2\theta+4)^2} \cdot 2\sec^2\theta d\theta = \int \frac{1}{(4(\tan^2\theta+1))^2} \cdot 2\sec^2\theta d\theta$$

$(4\sec^2\theta)^2$

Trig Sub

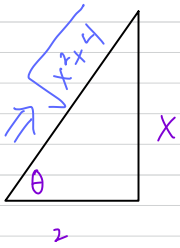
$x = 2\tan \theta$
$dx = 2\sec^2 \theta d\theta$

$$= \int \frac{1}{16 \sec^4 \theta} \cdot 2\sec^2 \theta d\theta = \frac{1}{8} \int \frac{1}{\sec^2 \theta} d\theta$$

Flip to Cosine

$$\tan \theta = \frac{x}{2} \Rightarrow \theta = \arctan \left(\frac{x}{2} \right)$$

2 copies left in denominator



$$= \frac{1}{8} \int \cos^2 \theta d\theta = \frac{1}{8} \int \frac{1 + \cos(2\theta)}{2} d\theta$$

Half-Angle Identity

$$= \frac{1}{16} \int 1 + \cos(2\theta) d\theta = \frac{1}{16} \left(\theta + \frac{\sin(2\theta)}{2} \right) + C$$

Double Angle Identity

$$= \frac{1}{16} \left[\arctan\left(\frac{x}{2}\right) + \left(\frac{x}{\sqrt{x^2+4}}\right) \left(\frac{2}{\sqrt{x^2+4}}\right) \right] + C$$

$$= \frac{1}{16} \left[\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4} \right] + C$$