Improper Integrals Definition for Type I and Type II

Type I: Unbounded Domain

Consider a function f that is continuous on an unbounded interval $[a, \infty)$.

$$\text{Define } \int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx \text{ provided the limit exists.}$$

Consider a function f that is continuous on an unbounded interval $(-\infty, b]$.

$$\text{Define } \int_{-\infty}^b f(x)\ dx = \lim_{s\to-\infty} \int_s^b f(x)\ dx \text{ provided the limit exists.}$$

Consider a function $f(x)$ that is continuous on the entire Real line.

$$\begin{aligned} \text{Define } \int_{-\infty}^\infty f(x)\ dx &= \int_{-\infty}^c f(x)\ dx + \int_c^\infty f(x)\ dx \quad \text{for some convenient choice of } c \\ &= \lim_{s\to-\infty} \int_s^c f(x)\ dx + \lim_{t\to\infty} \int_c^t f(x)\ dx \quad \text{provided BOTH converge} \end{aligned}$$

Type II: Unbounded Range

Consider $f(x)$ that is continuous on $[a, b)$, but discontinuous at $x = b$.

$$\text{Define } \int_a^b f(x)\ dx = \lim_{t\to b^-} \int_a^t f(x)\ dx \text{ provided the limit exists.}$$

Consider $f(x)$ that is continuous on $(a, b]$, but discontinuous at $x = a$.

$$\text{Define } \int_a^b f(x)\ dx = \lim_{s\to a^+} \int_s^b f(x)\ dx \text{ provided the limit exists.}$$

Consider $f(x)$ that is discontinuous at $x = c$ in (a, b).

$$\begin{aligned} \text{Define } \int_a^b f(x)\ dx &= \int_a^c f(x)\ dx + \int_c^b f(x)\ dx \quad \text{for some convenient choice of } c \\ &= \lim_{s\to c^-} \int_a^s f(x)\ dx + \lim_{t\to c^+} \int_t^b f(x)\ dx \quad \text{provided BOTH converge} \end{aligned}$$

- If the Limit exists, then we say the Improper Integral **Converges**.
- Otherwise, if the Limit Does Not Exist, then we say the Improper Integral **Diverges**.