Plan of Attack for Convergence of Infinite Series

Math 121

Given an Infinite Series $\sum_{n=1}^{\infty} a_n$, we can set a rough *Plan of Attack* to study its possible convergence. We need to decide which Convergence Test(s) applies. We will layer these options, depending on the question at hand. There is no Exact rule here, and while you can mix and match the application of Series convergence tests, this is only one suggestion to follow. We are not restating the Tests here, only making an organized plan.

Here is a rough strategic outline:

$\boxed{\text{Level 1:}}$ Any test that has a direct or more *"obvious"* application, meaning that the series terms are of an exact, certain form that make for a direct application of a Convergence Test, and therefore a quick conclusion as well.

- Geometric Series Test (GST)

Apply this test when the Series fits the Geometric form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$ where a (first term) and r (the Common Ratio Multiplier) are constants. Here every successive term is found by multiplying the previous term by the common ratio r. Geometric Series can be spotted by looking for *exponential-ish* pieces in both the numerator and denominator. Each piece can have different fixed base values with different changing variable exponentials.

Examples: $\sum_{n=1}^{\infty} \frac{7}{8^n}$ $\quad \sum_{n=1}^{\infty} 6^n$ $\quad \sum_{n=1}^{\infty} \frac{3^n}{7^n}$ $\quad \sum_{n=1}^{\infty} \frac{1}{7^{n-2}}$ $\quad \sum_{n=1}^{\infty} 8$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n\ 3^{n+1}}{2^{4n-1}}$

- p-Series

Apply this test when the Series fits the p-Series form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where the power p is a fixed Real number. Look for a fixed constant power and also a changing base in the variable n, essentially spotting a power function in the denominator.

Examples: $\sum_{n=1}^{\infty} \frac{1}{n^7}$ $\quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^3}$

- Ratio Test (RT) Test the Original Series right away

Apply this test to series where the terms involve Factorials and/or Exponential pieces.

Examples: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{n! \cdot n^n}$ $\quad \sum_{n=1}^{\infty} \frac{n^2}{8^n}$ $\quad \sum_{n=1}^{\infty} \frac{1}{6^n}$ $\quad \sum_{n=1}^{\infty} \frac{(\ln n)\ 3^n}{n^6 \cdot (n!)^2}$

• n^{th} Term Divergence Test (nTDT)

Apply this test **only** where the Terms of the series are NOT approaching 0, as $n \to \infty$. That is, $\lim_{n\to\infty} a_n \neq 0$. Use instincts on comparative size between our elementary functions to aid in the limit arguments. In the end, despite our instincts, we still need to *prove* the limiting size arguments, and that may involve a L'Hôpital's Rule. Certainly if the numerator is growing much faster than the denominator or there is no denominator, you might see a quick application of nTDT. Look for oddball-like series that don't fit another test.

Examples: $\sum_{n=1}^{\infty} \arctan n \qquad \sum_{n=1}^{\infty} 8 \qquad \sum_{n=2}^{\infty} \frac{n^2}{\ln n} \qquad \sum_{n=1}^{\infty} \frac{e^{2n}}{n^3} \qquad \sum_{n=1}^{\infty} n^3 + 8$

$$\sum_{n=1}^{\infty} \frac{n^3+8}{n^3+1} \qquad \sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right) \qquad \sum_{n=1}^{\infty} \left(1 + \arcsin\left(\frac{5}{n^2}\right)\right)^{n^2} \qquad \sum_{n=2}^{\infty} \frac{e^n}{\ln n}$$

• Arithmetic of Series (not a formal convergence test)

Apply these algebra arguments when your given series decomposes or splits into simpler pieces where separate tests more quickly apply.

Learn the three phrases:

Constant Multiple of a Convergent Series is Convergent.

Constant Multiple of a Divergent Series is Divergent.

The Sum of two Convergent Series is Convergent.

Examples: $\sum_{n=1}^{\infty} \frac{5n^2+\sqrt{n}}{n^4} = 5\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^{\frac{7}{2}}}$ Fill Details here

$\sum_{n=1}^{\infty} \frac{6}{n^9} + \frac{6^n}{9^n} = 6\sum_{n=1}^{\infty} \frac{1}{n^9} + \sum_{n=1}^{\infty} \left(\frac{6}{9}\right)^n$ Fill Details here

NOTE: we will now proceed to other tests under the assumption that the above Level 1 tests do not apply. In particular, if the Ratio Test does not apply; if it did, you already would have applied it.

$\boxed{\text{Level 2:}}$ Comparison Tests for $\oplus$ positive termed series, where we run external comparisons to simpler series, usually p-series or geometric series. Look for similar series in size, as $n \to \infty$, where $\sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{\infty} b_n$

• Direct Comparison Test (CT)

Apply this test when your given series is comparable to a simpler series, and you can **prove similarity by directly, quickly and simply bounding the terms**. This can be tricky to prove that the exact size arguments are in a helpful ordering as follows. We need:

Smaller than Converge is Converge or *Larger than Diverge is Diverge*

Examples: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^8+5}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^5+1}{n^6}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^6+7}$ $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^6+1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{6+9^n}$

• Limit Comparison Test (LCT)

Apply this test when your given series is comparable to a simpler series, and you can **prove similarity by studying the limiting value of the stack of the terms**. You will prove that the series *Share the same convergence behavior* without caring which terms are bigger or smaller. This test only cares which terms are *about* the same size as the comparison terms. LCT is helpful if there are stacks of polynomials pieces and there are several terms in (both) the numerator and/or denominator. In many ways this is a bit of a *lazy* test, comparative size-wise, but you do need to handle the limit. Divides-by-algebra can help.

Examples: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+9}{n^9+2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7n-4}{8n^7+6n^3-5}$ $\displaystyle\sum_{n=1}^{\infty} \frac{\text{polynomial}}{\text{polynomial}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{\text{constant}}{\text{polynomial}}$

NOTE: we will proceed to other tests under the assumption that the above Level 1 and Level 2 tests do not already apply. In particular, if the Ratio Test and Comparisons Tests does not immediately apply.

Level 3: for Alternating series or series with a mix of $\oplus/\ominus$ terms

This Level is more of a strategy involving organization and tests rather than just a straightforward application of a convergence test. There are two main tests, but the work **also depends on the given Question at hand.**

• Alternating Series Test (AST)

Apply this test when your series is Alternating and the non-alternating portion, b_n, of the terms in n are shrinking all the way to zero. Use when the question is asking about plain convergence of the original series. This result only yields a convergence result for the original series.

• Absolute Convergence Test (ACT)

Apply this test when you can show that the Absolute Series Converges, typically as in Level 2 above. This result also only yields a convergence result for the original series.

Study two separate Questions:

Question 1: Does the given series Converge or Diverge?

KEY: Based on the question, we can either use AST or ACT. We would only move to the Absolute Series is it is *helpful*, meaning that the Absolute Series will be proven Convergent. If the Absolute Series is Divergent, do not waste your time moving to the Absolute Series, under this Converge/Diverge question.

O.S. $\sum_{n=1}^{\infty} a_n$ $\xhookrightarrow{1^{st}}$ if "helpful" A.S. $\sum_{n=1}^{\infty} |a_n|$

$\swarrow$ OR 1^{st}

ACT vs AST Chart

fill $\downarrow$ work (A.S.)

fill $\downarrow$ work (O.S.)

$\boxed{\text{O.S. Converges}}$ by AST

$\boxed{\text{A.S Converges}}$ by __(fill here)__

OR $\swarrow$ $\Longleftarrow$

$\boxed{\text{O.S. Converges by Absolute Convergence Test ACT}}$

Examples:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6} \text{ AST or ACT} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^8+2} \text{ AST or ACT} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+3} \text{ AST only}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+8n-3)}{n^6+8} \text{ AST (messy) or ACT (nicer, can use LCT on the Absolute Series)}$$

NOTE: ACT is not just about moving to the Absolute Series; it's more about making a conclusion about the Original Series IF the Absolute Series Converges.

KEY: ACT says Absolute Convergence implies regular Convergence.

Question 2: Absolutely Convergent, Conditionally Convergent or Divergent?

KEY: Based on the question, we **MUST** move to the Absolute Series to determine (and prove) whether the Absolute Series is convergent, even if the Absolute Series is Divergent.

Here, we use the standard Absolute or Conditional Convergence Charts.

O.S. $\displaystyle\sum_{n=1}^{\infty} a_n \xrightarrow[\text{must}]{1^{st}}$ A.S. $\displaystyle\sum_{n=1}^{\infty} |a_n|$

A.C. Chart

no need to analyze O.S.

A.S. $\big\downarrow$ fill work

A.S Converges by (fill here)

O.S.Absolutely Convergent *by Definition*

O.S. $\displaystyle\sum_{n=1}^{\infty} a_n \xrightarrow[\text{must}]{1^{st}}$ A.S. $\displaystyle\sum_{n=1}^{\infty} |a_n|$

O.S. $\swarrow 2^{nd}$

C.C. Chart

O.S. $\big\downarrow$ fill work

O.S. Converges by AST

A.S. $\big\downarrow$ fill work

A.S Diverges by (fill here)

$\Rightarrow$ O.S. is **NOT** A.C.

$\Rightarrow$ must analyze O.S.

O.S. Conditionally Conv. *by Definition*

NOTE: Move to study the Absolute series if you are required to (meaning the question is Absolutely or Conditionally Convergent) OR if the Absolute Series is helpful (meaning the Absolute Series is Convergent) allowing for application of ACT.

Key: to study the Absolute series, with positive terms, the problem typically reduces to Level 2 (above) or Level 4 (below).

Examples: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6+9}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+7}$ $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+9)}{5n^9-3n+8}$ $\sum_{n=1}^{\infty} \frac{(-1)^n(n^3+5)}{n^4}$

Level 4: for any series where the above tests don't easily apply.

- Integral Test (IT)

Apply this test when the related function is not too hard to integrate. Used for postive-termed series. We often think of this as the last resort since Integration can be difficult or time consuming. This test is GOOD for series combining logarithms and polynomials. This test works well on series like $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, but a direct comparison is much faster. Feel free to list this test higher on your list of strategies, if you enjoy the long integrals.

Examples: $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Convergence Tests involving Limits

1. n^{th} Term Divergence Test: studies when $\lim_{n\to\infty} a_n \neq 0$ The value 0 is Inconclusive here.

2. Limit Comparison Test: studies the limit of the Ratio of terms $\lim_{n\to\infty} \frac{a_n}{b_n}$ for comparable series $\sum a_n \approx \sum b_n$. We choose the comparison series so that this limit is Finite and Non-Zero. This limit can be 1, but the limit cannot be 0 or ∞. If you get 0 or ∞ that usually, again, means you maybe chose the wrong comparison series or have an algebra mistake in the limit flip.

3. Ratio Test: studies the Ratio of successive terms $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ The value 1 is Inconclusive here. This limit can be 0 or it can be ∞, the opposite of the Limit Comparison Test.

4. Alternating Series: studies whether the terms of an alternating series $\sum(-1)^n b_n$ are shrinking to zero, that is, $\lim_{n\to\infty} b_n = 0$.

5. Integral Test: studies the convergence of the Improper Integral (Type I) of the Related function, which by definition involves a limit $\int_1^{\infty} f(x)\, dx = \lim_{t\to\infty} \int_1^t f(x)\, dx$

Next we will set a *Plan of Attack* for each Convergence Test.

Geometric Series Test

1. Given a Geometric Series, write out the first few terms $a + ar + ar^2 + \dots$

2. Pick off the a which equals the first term.

3. Pick off the Common Ratio Multiplier $\quad r = \dfrac{a_{n+1}}{a_n} = \dfrac{\text{next term}}{\text{previous term}}$

4. Measure the size of $|r|$ against the value 1.

5. Conclude Converge or Diverge. If asked, find the Sum=$\dfrac{a}{1-r}$

n^{th} Term Divergence Test

1. Study the size of the terms, $\lim\limits_{n\to\infty} a_n$.

2. Compare the limit with the value 0.

3. Conclude Divergence if the limit if NOT equal to 0. Otherwise, if the limit is equal to 0, STOP and find another test, the nTDT does not apply.

Integral Test

1. Given a series with positive terms, find the related function, replace the n terms with x.

2. Check the 3 pre-conditions (positive, continuous, decreasing) if required.

3. Compute the Improper Integral $\displaystyle\int_1^\infty f(x)\, dx$, and use L'Hôpital's Rule if needed on the Improper Limit finish.

4. Declare the Improper Integral to be Convergent or not (Finite or not?). Be clear.

5. Conclude Convergence or Divergence of the Original related Series. They share the same Convergence behavior. Parts 4/5 are two separate conclusions.

p-Series Test or p-Test

1. Given a p-series, pick off the power p

2. Measure p against the value 1.

3. Conclude Converge or Diverge

(Direct) Comparison Test

1. Pick a comparison series that is similar (yet simpler) to the given series, by ignoring non-dominant terms in both the numerator and the denominator.

2. Analyze the Comparison series fully. Make a clear conclusion of converge or diverge.

3. Directly compare the size of the terms of the original and the comparison series. Bound the Terms, measuring which terms are explicitly bigger or smaller.

4. Make final conclusion about the Original Series. Conclude converge or diverge as they share the same (convergence) behavior.

Continued: Setting a *Plan of Attack* for each Convergence Test.

Limit Comparison Test

Note: Starts the same as the Comparison Test.

1. Pick a comparison series that is similar (yet simpler) to the given series, by ignoring non-dominant terms in both the numerator and the denominator.

2. Analyze the Comparison series fully. Make a clear conclusion of converge or diverge.

3. Measure how the terms compare, by measuring the Limit of the stack of the terms. Look for a finite number. If it is not finite and non-zero, we either chose the wrong comparison series OR we simply made an algebra mistake, which is easy to do.

4. Mark Finite and Non-Zero to justify that the series are indeed *comparable*

5. Make final conclusion about the Original Series. Conclude converge or diverge as they share the same (convergence) behavior.

Alternating Series Test

1. Given an alternating Series $\sum(-1)^n b_n$, pick off the non-alternating piece, $\oplus$ terms $b_n > 0$

2. Check the two size conditions, that the terms are decreasing, all the way to zero.

3. Conclude Converge if the conditions are satisfied. Otherwise, STOP the AST and finish with the Divergence Test instead.

Absolute Convergence Test

1. Find the Absolute Series of a given series by dropping all negative signs.

2. Analyze the Absolute series fully. Usually involves Comparison tests or the Integral Test.

3. Conclude convergence (only one that's helpful) of the Absolute series. Make clear which test(s) used.

4. Conclude Convergence of original series if the Absolute Series converges. Otherwise, if the Absolute series diverges, STOP, the ACT will not apply.

Ratio Test

1. Given a series with a factorial and/or exponential term piece(s), study the limiting value of the Ratio of sucessive terms. The order does matter here. That is, study $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

2. Measure this limit L against the value 1.

3. Conclude Absolute Convergence or Divergence. If $L = 1$, the the test is Inconclusive. You can usually avoid this $L = 1$ issues if you do not apply the Ratio Test to series with only Logs and polynomial pieces.