

### Extra Examples of Sequences

Determine **and state** whether the following sequences **converges** or **diverges**. If it converges, compute its limit. Justify your answer. Do **not** just put down a number.

$$1. \left\{ \cos \left( \pi + \frac{1}{n} \right) \right\}_{n=1}^{\infty} \quad \hookrightarrow$$

$$\lim_{n \rightarrow \infty} \cos \left( \pi + \frac{1}{n} \right) = \cos \left( \lim_{n \rightarrow \infty} \pi + \frac{1}{n} \right) = \cos(\pi + 0) = \cos \pi = \boxed{-1} \quad \boxed{\text{Converges}}$$

Notice that we passed the limit inside the continuous cosine function.

$$2. \left\{ \frac{3n^2 - n + 2}{5n^2 + 7} \right\}_{n=1}^{\infty} \quad \hookrightarrow$$

$$\lim_{n \rightarrow \infty} \frac{3n^2 - n + 2}{5n^2 + 7} = \lim_{n \rightarrow \infty} \frac{3n^2 - n + 2}{5n^2 + 7} \cdot \left( \frac{1}{\frac{1}{n^2}} \right) = \lim_{n \rightarrow \infty} \frac{3 - \frac{1}{n} + \frac{2}{n^2}}{5 + \frac{7}{n^2}} = \boxed{\frac{3}{5}} \quad \boxed{\text{Converges}}$$

Or L'H Rule can also be used, but must switch to  $x$  variable.

$$3. \left\{ \sqrt{\frac{4n^2 + 5}{9n^2 - 2n + 1}} \right\}_{n=1}^{\infty} \quad \hookrightarrow$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt{\frac{4n^2 + 5}{9n^2 - 2n + 1}} &= \sqrt{\lim_{n \rightarrow \infty} \frac{4n^2 + 5}{9n^2 - 2n + 1}} = \sqrt{\lim_{n \rightarrow \infty} \frac{4n^2 + 5}{9n^2 - 2n + 1} \cdot \left( \frac{1}{\frac{1}{n^2}} \right)} \\ &= \sqrt{\lim_{n \rightarrow \infty} \frac{4 + \frac{5}{n^2}}{9 - \frac{2}{n} + \frac{1}{n^2}}} = \sqrt{\frac{4 + 0}{9 - 0 + 0}} = \sqrt{\frac{4}{9}} = \boxed{\frac{2}{3}} \quad \boxed{\text{Converges}} \end{aligned}$$

$$\begin{aligned}
4. \quad & \left\{ \left(1 - \frac{3}{n^2}\right)^{n^2} \right\}_{n=1}^{\infty} \hookrightarrow \\
& \lim_{n \rightarrow \infty} \left(1 - \frac{3}{n^2}\right)^{n^2} = \lim_{x \rightarrow \infty} \left(1 - \frac{3}{x^2}\right)^{x^2} = e^{\lim_{x \rightarrow \infty} \ln \left[ \left(1 - \frac{3}{x^2}\right)^{x^2} \right]} \\
& = e^{\lim_{x \rightarrow \infty} x^2 \ln \left(1 - \frac{3}{x^2}\right)} = e^{\lim_{x \rightarrow \infty} \frac{\ln \left(1 - \frac{3}{x^2}\right)}{\frac{1}{x^2}}} \\
& \stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{\left(\frac{1}{1 - \frac{3}{x^2}}\right) \left(\frac{6}{x^3}\right)}{-\frac{2}{x^3}}} = e^{\lim_{x \rightarrow \infty} \frac{1}{1 - \frac{3}{x^2}} \cdot (-3)} = \boxed{\frac{1}{e^3}} \quad \boxed{\text{Converges}}
\end{aligned}$$

$$\begin{aligned}
5. \quad & \left\{ \left(\frac{n}{n+1}\right)^n \right\}_{n=1}^{\infty} \hookrightarrow \\
& \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \lim_{x \rightarrow \infty} \left(\frac{x}{x+1}\right)^x = e^{\lim_{x \rightarrow \infty} \ln \left[ \left(\frac{x}{x+1}\right)^x \right]} \\
& = e^{\lim_{x \rightarrow \infty} x \ln \left(\frac{x}{x+1}\right)} = e^{\lim_{x \rightarrow \infty} \frac{\ln \left(\frac{x}{x+1}\right)}{\frac{1}{x}}} \\
& \stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{\left(\frac{x+1}{x}\right) \left(\frac{(x+1)(1) - x(1)}{(x+1)^2}\right)}{-\frac{1}{x^2}}} = e^{\lim_{x \rightarrow \infty} \frac{\left(\frac{x+1}{x}\right) \left(\frac{1}{(x+1)^2}\right)}{-\frac{1}{x^2}}} \\
& = e^{\lim_{x \rightarrow \infty} \left(\frac{-x}{x+1}\right)} \stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \left(\frac{-1}{1}\right)} = e^{-1} = \boxed{\frac{1}{e}} \quad \boxed{\text{Converges}}
\end{aligned}$$

$$6. \left\{ \left( \frac{n}{n+5} \right)^{2n+1} \right\}_{n=1}^{\infty} \hookrightarrow$$

$$\lim_{n \rightarrow \infty} \left( \frac{n}{n+5} \right)^{2n+1} \stackrel{(1^\infty)}{=} \lim_{x \rightarrow \infty} \left( \frac{x}{x+5} \right)^{2x+1} \stackrel{(1^\infty)}{=} e^{\lim_{x \rightarrow \infty} \ln \left[ \left( \frac{x}{x+5} \right)^{2x+1} \right]}$$

$$= e^{\lim_{x \rightarrow \infty} (2x+1) \ln \left( \frac{x}{x+5} \right)} \stackrel{(\infty \cdot 0)}{=} e^{\lim_{x \rightarrow \infty} \frac{\ln \left( \frac{x}{x+5} \right)}{\frac{1}{2x+1}}}$$

$$\stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{\left( \frac{x+5}{x} \right) \left( \frac{(x+5)(1) - x(1)}{(x+5)^2} \right)}{-\frac{2}{(2x+1)^2}}} = e^{\lim_{x \rightarrow \infty} \frac{\left( \frac{x+5}{x} \right) \left( \frac{5}{(x+5)^2} \right)}{-\frac{2}{(2x+1)^2}}}$$

$$= e^{\lim_{x \rightarrow \infty} \left( \frac{x+5}{x} \right) \left( \frac{5}{(x+5)^2} \right) \left( -\frac{(2x+1)^2}{2} \right)}$$

$$= e^{\lim_{x \rightarrow \infty} \left( \frac{-5(4x^2 + 4x + 1)}{2(x^2 + 5x)} \right)} = e^{\lim_{x \rightarrow \infty} \left( \frac{-5(4x^2 + 4x + 1)}{2(x^2 + 5x)} \right) \left( \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right)}$$

$$= e^{\lim_{x \rightarrow \infty} \left( \frac{-5 \left( 4 + \frac{4}{x} + \frac{1}{x^2} \right)}{2 \left( 1 + \frac{5}{x} \right)} \right)} = e^{-\frac{20}{2}} = e^{-10} = \boxed{\frac{1}{e^{10}}} \quad \boxed{\text{Converges}}$$

$$7. \left\{ \frac{(2n+1)!}{(2n)!} \right\} \hookrightarrow$$

$$\lim_{n \rightarrow \infty} \frac{(2n+1)!}{(2n)!} = \lim_{n \rightarrow \infty} \frac{(2n+1)(2n)(2n-1) \dots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2n)(2n-1)(2n-2) \dots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \lim_{n \rightarrow \infty} 2n+1 = \infty \quad \boxed{\text{Diverges}}$$

$$\text{OR } \lim_{n \rightarrow \infty} \frac{(2n+1)!}{(2n)!} = \lim_{n \rightarrow \infty} \frac{(2n+1)(2n)!}{(2n)!} = \lim_{n \rightarrow \infty} 2n+1 = \infty \quad \boxed{\text{Diverges}}$$