Math 121 Self-Assessment Quiz #9

Answer Key

- Please see the course webpage for the answer key.

1. (a) Use the MacLaurin Series representation for $f(x) = x^3 \sin\left(x^2\right)$ to

$$\textbf{Estimate} \quad \int_0^1 x^3 \sin\left(x^2\right)\, dx \quad \text{with error less than } \frac{1}{100}.$$

Justify in words that your error is indeed less than $\frac{1}{100}$.

$$\int_0^1 x^3 \sin\left(x^2\right)\, dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^2\right)^{2n+1}}{(2n+1)!}\, dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\, dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(2n+1)!}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+6}}{(2n+1)!(4n+6)}\Bigg|_0^1$$

$$= \frac{x^6}{6} - \frac{x^{10}}{3!(10)} + \frac{x^{14}}{5!(14)} - \ldots \Bigg|_0^1 = \frac{1}{6} - \frac{1}{60} + \frac{1}{1680} + \ldots - (0 - 0 + 0 - \ldots)$$

$$\approx \frac{1}{6} - \frac{1}{60} = \frac{10}{60} - \frac{1}{60} = \frac{9}{60} = \boxed{\frac{3}{20}} \leftarrow \text{estimate}$$

Using the Alternating Series Estimation Theorem, if we approximate the actual sum with only the first two terms, the error from the actual sum will be *at most* the absolute value of the next (first neglected) term, $\frac{1}{1680}$. Here $\frac{1}{1680} < \frac{1}{100}$ as desired.

(b) Estimate $\frac{1}{\sqrt{e}}$ with error less than $\frac{1}{100}$ Justify.

Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$$\frac{1}{\sqrt{e}} = e^{-\frac{1}{2}} = 1 - \frac{1}{2} + \frac{\left(-\frac{1}{2}\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)^3}{3!} + \frac{\left(-\frac{1}{2}\right)^4}{4!} + \ldots = 1 - \frac{1}{2} + \frac{\frac{1}{4}}{2!} - \frac{\frac{1}{8}}{3!} + \frac{\frac{1}{16}}{4!} + \ldots$$

$$= 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} + \ldots \approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} = \frac{48}{48} - \frac{24}{48} + \frac{6}{48} - \frac{1}{48} = \boxed{\frac{29}{48}} \leftarrow \text{estimate}$$

Using ASET we can estimate the full sum using only the first four terms with error *at most* (the first neglected term in absolute value) $\frac{1}{384} < \frac{1}{100}$ as desired.

2. Find the **sum** for each of the following series.

(a) $1 - 2 + \frac{4}{2!} - \frac{8}{3!} + \frac{16}{4!} - \frac{32}{5!} + \ldots = 1 + (-2) + \frac{(-2)^2}{2!} + \frac{(-2)^3}{3!} + \frac{(-2)^4}{4!} + \frac{(-2)^5}{5!} + \ldots = \boxed{e^{-2}}$

(b) $$\frac{1}{3\pi} - \frac{1}{18\pi^2} + \frac{1}{81\pi^3} - \frac{1}{324\pi^4} + \ldots = \frac{1}{3\pi} - \frac{1}{2\cdot 9\pi^2} + \frac{1}{3\cdot(27)\pi^3} - \frac{1}{4\cdot(81)\pi^4} + \ldots$$

$$= \frac{1}{3\pi} - \frac{1}{2(3\pi)^2} + \frac{1}{3(3\pi)^3} - \frac{1}{4(3\pi)^4} + \ldots = \boxed{\ln\left(1+\frac{1}{3\pi}\right)}$$

(c) $$\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 9)^n}{2^{n+1}\ n!} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 9)^n}{2^n\ n!} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\left(-\frac{\ln 9}{2}\right)^n}{n!} = \frac{1}{2}e^{-\frac{\ln 9}{2}} = \frac{1}{2}e^{\ln\left(9^{-\frac{1}{2}}\right)}$$

$$= \frac{1}{2}\left(9^{-\frac{1}{2}}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{9}}\right) = \frac{1}{2}\left(\frac{1}{3}\right) = \boxed{\frac{1}{6}}$$

(d) $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \arctan(1) = \boxed{\frac{\pi}{4}}$$

(e) $$-\frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \ldots = \cos\pi - 1 = -1 - 1 = \boxed{-2}$$

note: this is the formula for $\cos(\pi)$ but we're missing the first term of 1

(f) $$\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \frac{1}{189\sqrt{3}} + \ldots = \frac{1}{\sqrt{3}} - \frac{1}{3(3\sqrt{3})} + \frac{1}{5(9\sqrt{3})} - \frac{1}{7(27\sqrt{3})} + \ldots$$

$$= \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \ldots = \arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{\pi}{6}}$$

(g) $$\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{9^n\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{3^{2n}\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{3}\right)^{2n}}{(2n+1)!} \cdot \frac{\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)}$$

$$= \frac{3}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{3}{\pi}\sin\left(\frac{\pi}{3}\right) = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3\sqrt{3}}{2\pi}}$$

(h) $$\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{2^{4n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(2^2)^{2n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{4^{2n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$$