Math 121 Self-Assessment Quiz #7

Answer Key

• Please see the course webpage for the answer key.

1. In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

a. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2-1}$

First, we show the absolute series $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-1}$ is divergent using CT.

Bound the terms $\displaystyle\frac{n}{n^2-1} > \frac{n}{n^2} = \frac{1}{n}$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent Harmonic p-series with $p = 1$.

Therefore the Absolute Series is Divergent by CT.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

•$b_n = \dfrac{n}{n^2-1} > 0$ for $n \geq 2$

• $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2-1} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1-\frac{1}{n^2}} = 0$

•$b_{n+1} < b_n$ since we can show the derivative of the related function is negative, hence the terms are decreasing

Consider $f(x) = \dfrac{x}{x^2-1}$ with $f'(x) = \dfrac{-x^2-1}{(x^2-1)^2} < 0$.

Therefore, the original series converges by the Alternating Series Test. Finally, we can conclude the original series is Conditionally Convergent

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(Note you can also use LCT to show the Absolute Series above diverges.)

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n}{n^2-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n^2}} = 1$ which is finite and non-zero.

Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then the absolute series also diverges by LCT.

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b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3\ (3n)!\ \ln n}{(n!)^4\ 2^{4n}\ n^n}$

Try Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}\ (n+1)^3\ (3(n+1))!\ \ln(n+1)}{((n+1)!)^4\ 2^{4(n+1)}(n+1)^{n+1}}}{\dfrac{(-1)^n\ n^3\ (3n)!\ \ln n}{(n!)^4\ 2^{4n}n^n}}\right|$$

$$= \lim_{n\to\infty}\left(\frac{(3n+3)!}{(3n)!}\right)\left(\frac{n^n}{(n+1)^{n+1}}\right)\left(\frac{(n!)^4}{((n+1)!)^4}\right)\left(\frac{2^{4n}}{2^{4n+4}}\right)\left(\frac{(n+1)^3}{n^3}\right)\left(\frac{\ln(n+1)}{\ln n}\right)$$

$$\overset{(*)}{=} \lim_{n\to\infty}\left(\frac{(3n+3)(3n+2)(3n+1)(3n)!}{(3n)!}\right)\left(\frac{n^n}{(n+1)^n(n+1)}\right)\left(\frac{(n!)^4}{(n+1)^4(n!)^4}\right)\left(\frac{2^{4n}}{2^{4n}\ 2^4}\right)\left(\frac{n+1}{n}\right)^3(1)$$

$$= \lim_{n\to\infty}\left(\frac{(3n+3)(3n+2)(3n+1)}{(n+1)^5\ 2^4}\right)\left(\frac{n^n}{(n+1)^n}\right)\left(1+\frac{1}{n}\right)^3$$

$$= \lim_{n\to\infty}\left(\frac{3(n+1)(3n+2)(3n+1)}{(n+1)^5\ 16}\right)\left(\frac{n^n}{(n+1)^n}\right)(1)$$

$$= \lim_{n\to\infty}\left(\frac{3(3n+2)(3n+1)}{16\ (n+1)^4}\right)\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{3(9n^2+9n+2}{16\ (n^2+2n+1)(n+1)^2}\right)\left(\frac{1}{e}\right)$$

$$= \lim_{n\to\infty}\left(\frac{3\left(9+\dfrac{9}{n}+\dfrac{2}{n^2}\right)}{16\left(1+\dfrac{2}{n}+\dfrac{1}{n}\right)}\right)\left(\frac{1}{(n+1)^2}\right)\left(\frac{1}{e}\right)$$

$$= \lim_{n\to\infty}\left(\frac{27}{16}\right)\left(\frac{1}{e}\right)\cdot 0 = 0 < 1$$

Therefore the original series $\boxed{\text{Converges Absolutely by the Ratio test}}$.

Here, from above,

$$(*) = \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}\overset{\frac{\infty}{\infty}}{} = \lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{x+1}}{\dfrac{1}{x}} = \lim_{x\to\infty}\frac{x}{x+1} \overset{\frac{\infty}{\infty}}{} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{1}{1} = 1$$

c. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n^5+6}{n^9+5\sqrt{n}+9}$

First, we show the absolute series $\displaystyle\sum_{n=1}^{\infty}\frac{3n^5+6}{n^9+5\sqrt{n}+9}$ is convergent using LCT.

We see that $\displaystyle\sum_{n=1}^{\infty}\frac{3n^5+6}{n^9+5\sqrt{n}+9} \approx \sum_{n=1}^{\infty}\frac{n^5}{n^9} = \sum_{n=1}^{\infty}\frac{1}{n^4}$ which is a convergent p-series $p = 4 > 1$.

Check: $\lim_{n\to\infty} \dfrac{\dfrac{3n^5+6}{n^9+5\sqrt{n}+9}}{\dfrac{1}{n^4}} = \lim_{n\to\infty} \dfrac{3n^9+6n^4}{n^9+5\sqrt{n}+9} \cdot \dfrac{\left(\dfrac{1}{n^9}\right)}{\left(\dfrac{1}{n^9}\right)} = \lim_{n\to\infty} \dfrac{3+\dfrac{6}{n^5}}{1+\dfrac{5}{n^{\frac{17}{2}}}+\dfrac{9}{n^9}} = 3$

which is finite and non-zero. Therefore, these two series share the same behavior. Since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges, then the absolute series also converges by LCT. As a result, since the absolute series converges, the original series converges by ACT. Finally, we have $\boxed{\text{Absolute Convergence}}$.

d. $\sum_{n=1}^{\infty} \dfrac{7+n^2}{5n^2-n+14}$

First examine the limiting value of the terms:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7+n^2}{5n^2-n+14} = \lim_{n\to\infty} \frac{7+n^2}{5n^2-n+14} \cdot \frac{\left(\dfrac{1}{n^2}\right)}{\left(\dfrac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\dfrac{7}{n^2}+1}{5-\dfrac{1}{n}+\dfrac{14}{n^2}} = \frac{1}{5} \neq 0$$

Therefore, the series $\boxed{\text{Diverges by the } n^{th} \text{ Term Divergence Test.}}$