Math 121 Self-Assessment Quiz #6

Answer Key

- Please see the course webpage for the answer key.

1. Find the **sum** of the following series $\sum_{n=1}^{\infty}(-1)^n \dfrac{6^{n+1}}{5^{3n-1}}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n\ 6^{n+1}}{5^{3n-1}} = -\frac{6^2}{5^2} + \frac{6^3}{5^5} - \frac{6^4}{5^8} + \dots$$

Here we have a geometric series with $a = -\dfrac{36}{25}$ and $r = -\dfrac{6}{5^3} = -\dfrac{6}{125}$. Note, it does converge since $|r| = \left|-\dfrac{6}{125}\right| = \dfrac{6}{125} < 1$.

As a result, the sum is given by SUM$=\dfrac{a}{1-r} = \dfrac{-\frac{36}{25}}{1-\left(-\frac{6}{125}\right)} = \dfrac{-\frac{36}{25}}{\frac{131}{125}} = -\dfrac{36}{25}\cdot\dfrac{125}{131} = \boxed{-\dfrac{180}{131}}$

2. Use the **Integral Test** to **determine** and **state** whether the series $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^2}$ converges or diverges. Justify all of your work.

Consider the related function $f(x) = \dfrac{\ln x}{x^2}$ with

1. $f(x)$ continuous for all $x > 0$
2. $f(x)$ positive for $x > 1$
3. $f(x)$ decreasing because $f'(x) = \dfrac{x^2\left(\frac{1}{x}\right) - \ln x(2x)}{(x^2)^2} = \dfrac{1-2\ln x}{x^3} < 0$ when $x > e^{\frac{1}{2}}$.

Check the improper integral

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{t\to\infty} \int_1^t (\ln x)x^{-2}\, dx \overset{\text{IBP}}{=} \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_1^t + \int_1^t x^{-2}\, dx$$

$$= \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_1^t - \frac{1}{x}\Big|_1^t = \lim_{t\to\infty} -\frac{\ln t}{t}^{\frac{\infty}{\infty}} + \frac{\ln 1}{1} - \left(\frac{1}{t} - \frac{1}{1}\right)$$

$$\overset{\text{L'H}}{=} \lim_{t\to\infty} -\frac{\left(\frac{1}{t}\right)}{1} + 0 - 0 + 1 = 1$$

The improper integral converges, and therefore the original series $\boxed{\text{Converges}}$ by the Integral Test (IT).

IBP: $\boxed{\begin{array}{ll} u = \ln x & dv = x^{-2}dx \\ du = \frac{1}{x}dx & v = -\frac{1}{x} \end{array}}$

3. In each case determine whether the given series **converges**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^7+2n+3} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^4}$

Bound the terms $\displaystyle\frac{n^3}{n^7+2n+3} < \frac{n^3}{n^7} = \frac{1}{n^4}$

and the comparison series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p-series with $p = 4 > 1$.

Finally, the original series (O.S.) $\boxed{\text{Converges by CT}}$.

b. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{2}{n^2}\right)^{n^2}$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$ because

$$\lim_{n\to\infty} \left(1-\frac{2}{n^2}\right)^{n^2 1^\infty} \quad \lim_{x\to\infty} \left(1-\frac{2}{x^2}\right)^{x^2} = \lim_{x\to\infty} e^{\ln\left(\left(1-\frac{2}{x^2}\right)^{x^2}\right)}$$

$$= e^{\lim\limits_{x\to\infty} \ln\left(\left(1-\frac{2}{x^2}\right)^{x^2}\right)} = e^{\lim\limits_{x\to\infty} x^2 \ln\left(1-\frac{2}{x^2}\right)} = e^{\lim\limits_{x\to\infty} \frac{\ln\left(1-\frac{2}{x^2}\right)}{\frac{1}{x^2}}\, \frac{0}{0}}$$

$$\overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty} \frac{\frac{1}{1-\frac{2}{x^2}}\left(\frac{4}{x^3}\right)}{-\frac{2}{x^3}}} = e^{\lim\limits_{x\to\infty} \frac{1}{1-\frac{2}{x^2}}(-2)} = e^{-2} \neq 0$$

c. $\displaystyle\sum_{n=4}^{\infty} \frac{n}{n^{\frac{3}{2}}-6} \approx \sum_{n=4}^{\infty} \frac{n}{n^{\frac{3}{2}}} = \sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}}$

Bound the terms $\displaystyle\frac{n}{n^{\frac{3}{2}}-6} > \frac{n}{n^{\frac{3}{2}}} = \frac{1}{\sqrt{n}}$ and the comparison series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a divergent p-series with $p = \dfrac{1}{2} < 1$. Finally, the Original Series $\boxed{\text{Diverges by CT}}$.

d. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^4+n-1}{n^5+n^2+3}$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{3n^4+n-1}{n^5+n^2+3} \approx \sum_{n=1}^{\infty} \frac{n^4}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent Harmonic p-series with $p = 1$.

Next,

Check: $\lim_{n\to\infty} \dfrac{\dfrac{3n^4+n-1}{n^5+n^2+3}}{\dfrac{1}{n}} = \lim_{n\to\infty} \dfrac{3n^5+n^2-n}{n^5+n^2+3} \dfrac{\left(\dfrac{1}{n^5}\right)}{\left(\dfrac{1}{n^5}\right)} = \lim_{n\to\infty} \dfrac{3+\dfrac{1}{n^3}-\dfrac{1}{n^4}}{1+\dfrac{1}{n^3}+\dfrac{3}{n^5}} = 3$ which is finite and non-zero $(0 < 3 < \infty)$.

Therefore, these two series share the same convergence behavior, and the O.S. is also $\boxed{\text{Divergent by Limit Comparison Test (LCT)}}$.

e. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{n^7+9}$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{n^7+9} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ which is the convergent p-series with $p = 5 > 1$. Next,

Check: $\lim_{n\to\infty} \dfrac{\dfrac{n^2+\sqrt{n}}{n^7+9}}{\dfrac{1}{n^5}} = \lim_{n\to\infty} \dfrac{n^7+n^{\frac{11}{2}}}{n^7+9} \dfrac{\left(\dfrac{1}{n^7}\right)}{\left(\dfrac{1}{n^7}\right)} = \lim_{n\to\infty} \dfrac{1+\dfrac{1}{n^{\frac{3}{2}}}}{1+\dfrac{9}{n^7}} = 1$ which is finite and non-zero $(0 < 1 < \infty)$.

Therefore, these two series share the same convergence behavior, and the O.S. is also $\boxed{\text{Convergent by Limit Comparison Test (LCT)}}$.

f. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{5^n (n!)^2}$

Try Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(2(n+1))!}{5^{n+1}\ ((n+1)!)^2}}{\dfrac{(-1)^n (2n)!}{5^n\ (n!)^2}} \right| = \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{(n!)^2}{((n+1)!)^2}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \cdot \frac{5^n}{5^n \cdot 5} \cdot \frac{(n!)^2}{(n+1)^2 (n!)^2} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{5(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{4n^2+6n+2}{5(n^2+2n+1)} \frac{\left(\dfrac{1}{n^2}\right)}{\left(\dfrac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{4+\dfrac{6}{n}+\dfrac{2}{n^2}}{5\left(1+\dfrac{2}{n}+\dfrac{1}{n^2}\right)} = \frac{4}{5} < 1$$

The original series is $\boxed{\text{Absolutely Convergent by the Ratio Test}}$ and therefore $\boxed{\text{Convergent by ACT}}$.