Math 121 Self-Assessment Quiz #5

Answer Key

1. Compute $\displaystyle\int_0^{e^5} \frac{1}{x\,(25+(\ln x)^2)}\,dx = \lim_{t\to 0^+}\int_t^5 \frac{1}{x(25+(\ln x)^2)}\,dx = \lim_{t\to 0^+}\int_{\ln t}^5 \frac{1}{25+w^2}\,dw$

$$= \lim_{t\to 0^+}\frac{1}{5}\arctan\left(\frac{w}{5}\right)\Big|_{\ln t}^{5} = \lim_{t\to 0^+}\frac{1}{5}\left(\arctan \lim_{t\to 0^+}\left(\frac{5}{5}\right) - \arctan\left(\frac{\ln t}{5}\right)\right)$$

$$= \lim_{t\to 0^+}\frac{1}{5}\left(\arctan(1) - \arctan\left(\frac{\ln t}{5}\right)\right) = \frac{1}{5}\left(\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{5}\left(\frac{\pi}{4}\right) = \boxed{\frac{3\pi}{20}} \text{ Converges}$$

Note: $\ln t$ approaches $-\infty$ as t approaches 0 positively. Then $\arctan \ln t$ approaches $-\dfrac{\pi}{2}$ as the input $\ln t$ approaches $-\infty$.

$w = \ln x$
$dw = \frac{1}{x}\,dx$

$x = t \Rightarrow w = \ln t$
$x = e^5 \Rightarrow w = \ln(e^5) = 5$

2. Compute $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\tan x}{(\ln(\cos x))^2}\,dx = \lim_{t\to\frac{\pi}{2}^-}\int_{\frac{\pi}{3}}^{t} \frac{\tan x}{(\ln(\cos x))^2}\,dx$

$$= \lim_{t\to\frac{\pi}{2}^-} -\int_{-\ln 2}^{\ln\cos t}\frac{1}{w^2}\,dw = \lim_{t\to\frac{\pi}{2}^-} -\left(-\frac{1}{w}\right)\Big|_{-\ln 2}^{\ln\cos t} = \lim_{t\to\frac{\pi}{2}^-}\frac{1}{w}\Big|_{-\ln 2}^{\ln\cos t}$$

$$= \lim_{t\to\frac{\pi}{2}^-}\frac{1}{\ln\cos t} - \left(\frac{1}{-\ln 2}\right) = 0 + \frac{1}{\ln 2} = \boxed{\frac{1}{\ln 2}} \text{ Converges}$$

because as $t \to \frac{\pi}{2}^-$ then $\cos t \to 0^+$ and then recall that the natural log approaches $-\infty$ as the input approaches 0^+. Finally $\frac{1}{\ln\cos t}$ approaches $\frac{1}{-\infty} = 0$

$w = \ln(\cos x)$
$dw = \frac{-\sin x}{\cos x}\,dx$
$dw = -\tan x\,dx$
$-dw = \tan x\,dx$

$x = \frac{\pi}{3} \Rightarrow w = \ln\left(\cos\left(\frac{\pi}{3}\right)\right) = \ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2$
$x = t \Rightarrow w = \ln(\cos t)$

3. Compute $\displaystyle\int_0^3 \frac{1}{\sqrt{x}(x+1)}\,dx = \lim_{t\to 0^+}\int_t^3 \frac{1}{\sqrt{x}((\sqrt{x})^2+1)}\,dx = \lim_{t\to 0^+} 2\int_{\sqrt{t}}^{\sqrt{3}}\frac{1}{w^2+1}\,dw$

$$= \lim_{t\to 0^+} 2\arctan w\Big|_{\sqrt{t}}^{\sqrt{3}} = \lim_{t\to 0^+} 2\left(\arctan\sqrt{3} - \arctan\sqrt{t}\right) = 2\left(\frac{\pi}{3} - 0\right) = \boxed{\frac{2\pi}{3}} \text{ Converges}$$

$$w = \sqrt{x}$$

$$dw = \frac{1}{2\sqrt{x}}\ dx$$

$$2dw = \frac{1}{\sqrt{x}}\ dx$$

$$x = t \Rightarrow w = \sqrt{t}$$
$$x = 3 \Rightarrow w = \sqrt{3}$$

4. Compute $\displaystyle\int_1^7 \frac{15-x}{x^2-6x-7}\ dx = \int_1^7 \frac{15-x}{(x-7)(x+1)}\ dx = \lim_{t\to 7^-} \int_1^t \frac{15-x}{(x-7)(x+1)}\ dx$

$$\overset{\text{PFD}}{=} \lim_{t\to 7^-} \int_1^t \frac{1}{x-7} - \frac{2}{x+1}\ dx = \lim_{t\to 7^-} \ln|x-7| - 2\ln|x+1|\Big|_1^t$$

$$= \lim_{t\to 7^-} \ln|t-7| - 2\ln 8 - (\ln|-6| - 2\ln 2) = \lim_{t\to 7^-} \ln|t-7| - 2\ln 8 - \ln 6 + 2\ln 2$$

$= -\infty - 2\ln 8 - \ln 6 + 2\ln 2 = \boxed{-\infty}$ Diverges

Partial Fractions Decomposition:

$$\frac{15-x}{(x-7)(x+1)} = \frac{A}{x-7} + \frac{B}{x+1}$$

Clearing the denominator yields:

$15 - x = A(x+1) + B(x-7)$
$15 - x = (A+B)x + A - 7B$
so that $A + B = -1$, and $A - 7B = 15$
Solve for $A = 1$, and $B = -2$

5. Compute $\displaystyle\int_0^1 \frac{1}{e^x - e^{-x}}\ dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{e^x - \frac{1}{e^x}}\ dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\left(\frac{e^{2x}-1}{e^x}\right)}\ dx$

$$= \lim_{t\to 0^+} \int_t^1 \frac{e^x}{e^{2x}-1}\ dx = \lim_{t\to 0^+} \int_t^1 \frac{e^x}{(e^x)^2-1}\ dx$$

$$= \lim_{t\to 0^+} \int_{e^t}^e \frac{1}{w^2-1}\ dw = \lim_{t\to 0^+} \int_{e^t}^e \frac{1}{(w-1)(w+1)}\ dw$$

$$\overset{\text{PFD}}{=} \lim_{t\to 0^+} \int_{e^t}^e \frac{\frac{1}{2}}{w-1} - \frac{\frac{1}{2}}{w+1}\ dw$$

$$= \lim_{t\to 0^+} \frac{1}{2}\ln|w-1| - \frac{1}{2}\ln|w+1|\Big|_{e^t}^e = \lim_{t\to 0^+} \frac{1}{2}\ln|e-1| - \frac{1}{2}\ln|e+1| - \left(\frac{1}{2}\ln|e^t-1| - \frac{1}{2}\ln|e^t+1|\right)$$

$$= \lim_{t\to 0^+} \frac{1}{2}\ln|e-1| - \frac{1}{2}\ln|e+1| - \frac{1}{2}\ln|e^t-1| + \frac{1}{2}\ln 2 = \frac{1}{2}\ln|e-1| - \frac{1}{2}\ln|e+1| - (-\infty) + \frac{1}{2}\ln 2 = \boxed{\infty}$$

Diverges

$$w = e^x$$

$$dw = e^x\ dx$$

$$x = t \Rightarrow w = e^t$$

$$x = 1 \Rightarrow w = e$$

Partial Fractions Decomposition:

$$\frac{1}{(w-1)(w+1)} = \frac{A}{w-1} + \frac{B}{w+1}$$

Clearing the denominator yields:

$1 = A(w+1) + B(w-1)$
$1 = (A+B)w + A - B$
so that $A + B = 0,$ and $A - B = 1$
Solve for $A = \frac{1}{2},$ and $B = -\frac{1}{2}$

6. Consider the sequence $\left\{\left(\frac{n}{n+1}\right)^n\right\}_{n=1}^{\infty}$. Compute $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n =$

We switch to the variable x and the related function $f(x) = \left(\frac{x}{x+1}\right)^x$ in order to apply L'H Rule.

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{x^{1^{\infty}}} = \lim_{x\to\infty} e^{\ln\left(\left(\frac{x}{x+1}\right)^x\right)}$$

$$= e^{\lim_{x\to\infty}\ln\left[\left(\frac{x}{x+1}\right)^x\right]} = e^{\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)^{\infty\cdot 0}} = e^{\lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)^{\frac{0}{0}}}{\frac{1}{x}}}$$

$$\overset{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{\left(\frac{1}{\frac{x}{x+1}}\right)\left(\frac{(x+1)(1)-x(1)}{(x+1)^2}\right)}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty}\left(\frac{x+1}{x}\right)\left(\frac{1}{(x+1)^2}\right)(-x^2)}$$

$$= e^{\lim_{x\to\infty} -\frac{x}{x+1}^{\frac{\infty}{\infty}}} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty} -\frac{1}{1}} = \boxed{e^{-1}}$$ and the sequence converges to $\frac{1}{e}$.