Math 121 Self-Assessment Quiz #2

Answer Key

- Please see the course webpage for the answer key.

1. Compute $\int_3^{3\sqrt{3}} \frac{1}{\sqrt{36-x^2}} + \frac{1}{9+x^2}\, dx = \arcsin\left(\frac{x}{6}\right)\Big|_3^{3\sqrt{3}} + \frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_3^{3\sqrt{3}}$

$$= \arcsin\left(\frac{3\sqrt{3}}{6}\right) - \arcsin\left(\frac{3}{6}\right) + \frac{1}{3}\arctan\left(\frac{3\sqrt{3}}{3}\right) - \frac{1}{3}\arctan\left(\frac{3}{3}\right)$$

$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right) + \frac{1}{3}\arctan\left(\sqrt{3}\right) - \frac{1}{3}\arctan(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{3}\right) - \frac{1}{3}\left(\frac{\pi}{4}\right) = \frac{\pi}{3} - \frac{\pi}{6} + \frac{\pi}{9} - \frac{\pi}{12} = \frac{12\pi}{36} - \frac{6\pi}{36} + \frac{4\pi}{36} - \frac{3\pi}{36} = \boxed{\frac{7\pi}{36}}$$

2. Compute $\int \frac{4}{(1+x^2)(1+(\arctan x)^2)}\, dx = 4\int \frac{1}{1+w^2}\, dw$

$$= 4\arctan w + C = \boxed{4\arctan(\arctan x) + C}$$

$w = \arctan x$
$dw = \frac{1}{1+x^2}\, dx$

3. Compute $\int_0^{\ln\sqrt{2}} \frac{e^x}{\sqrt{4-e^{2x}}}\, dx = \int_0^{\ln\sqrt{2}} \frac{e^x}{\sqrt{4-(e^x)^2}}\, dx$

$$= \int_1^{\sqrt{2}} \frac{1}{\sqrt{4-u^2}}\, du = \arcsin\left(\frac{u}{2}\right)\Big|_1^{\sqrt{2}} = \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi}{12} - \frac{2\pi}{12} = \boxed{\frac{\pi}{12}}$$

$u = e^x$	$x = 0 \Rightarrow u = e^0 = 1$
$du = e^x\, dx$	$x = \ln\sqrt{2} \Rightarrow u = e^{\ln\sqrt{2}} = \sqrt{2}$

4. Compute $\int_0^{\ln 3} \sinh(2x)\, dx = \frac{1}{2}\cosh(2x)\Big|_0^{\ln 3} = \frac{1}{2}\left(\cosh(2\ln 3) - \cosh 0\right)$

$$= \frac{1}{2}\left(\left(\frac{e^{2\ln 3} + e^{-2\ln 3}}{2}\right) - 1\right) = \frac{1}{2}\left(\left(\frac{e^{\ln(3^2)} + e^{\ln(3^{-2})}}{2}\right) - 1\right)$$

$$= \frac{1}{2}\left(\frac{9+\frac{1}{9}}{2} - 1\right) = \frac{1}{2}\left(\frac{\left(\frac{82}{9}\right)}{2} - 1\right) = \frac{1}{2}\left(\frac{41}{9} - 1\right) = \frac{1}{2}\left(\frac{32}{9}\right) = \boxed{\frac{16}{9}}$$

5. (a) Use implicit differentiation to **PROVE** that $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$

If $y = \arctan x$, then $\tan y = x$. Implicitly differentiate both sides,

$\dfrac{d}{dx}(\tan y) = \dfrac{d}{dx}(x)$.

Then $\sec^2 y\,\dfrac{dy}{dx} = 1$.

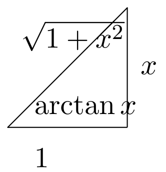
Solve for $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}$

Here we used the trig. identity to finish

OR finish

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\sec^2(\arctan x)} = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$$

using the trig. from the triangle



(b) From part (a) we now know that $\dfrac{1}{1+x^2}\,dx = \arctan x + C$. Use this fact **and integration** to **PROVE** that $\displaystyle\int \frac{1}{3+x^2}\,dx = \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$

$$\int \frac{1}{3+x^2}\,dx = \int \frac{1}{3\left(1+\frac{x^2}{3}\right)}\,dx = \frac{1}{3}\int \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx$$

$$= \frac{\sqrt{3}}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{\sqrt{3}}\arctan u + C = \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

Standard u substitution to simplify:

$$u = \frac{x}{\sqrt{3}}$$
$$du = \frac{1}{\sqrt{3}}\,dx$$
$$\sqrt{3}du = dx$$

Note: **OR** you can also do a trig. substitution here (in the future).

6. Use implicit differentiation to **PROVE** that $\dfrac{d}{dx}\sin^{-1}(5x) = \dfrac{5}{\sqrt{1-25x^2}}$

Let $y = \sin^{-1}(5x)$. Then $\sin y = 5x$. Implicitly differentiate both sides,

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(5x).$$

Then $\cos y \, \frac{dy}{dx} = 5$ Solve for $\frac{dy}{dx} = \frac{5}{\cos y} = \frac{5}{\sqrt{1 - \sin^2 y}} = \frac{5}{\sqrt{1 - 25x^2}}$