# Exam 3 Spring 2023 Answer Key

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (5x+1)^n}{(5n+1)\cdot 4^n}$ Ratio Test

Converges by Ratio Test when

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(5x+1)^{n+1}}{(5(n+1)+1)\cdot 4^{n+1}}}{\frac{\cancel{(-1)^n}(5x+1)^n}{(5n+1)\quad 4^n}}\right| = \lim_{n\to\infty}\underbrace{\left|\frac{(5x+1)^{n+1}}{\cancel{(5x+1)^n}}\right|}_{|5x+1|}\cdot\underbrace{\cancel{\left(\frac{5n+1}{5n+6}\right)}}_{1}\cdot\frac{\cancel{4^n}}{\underbrace{\cancel{4^{n+1}}}_{\cancel{4^n}\cdot 4}} = \frac{|5x+1|}{4} < 1$$

$$\frac{|5x+1|}{4} < 1 \Rightarrow |5x+1| < 4 \Rightarrow \underset{-1}{-4} < \underset{-1}{5x+1} < \underset{-1}{4} \Rightarrow -5 < 5x < 3 \Rightarrow -1 < x < \frac{3}{5}$$

Manually Check Convergence at Endpoints

Take $x = -1$. Series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n[5(-1)+1]^n}{(5n+1)\cdot 4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-4)^n}{(5n+1)\cdot 4^n} = \sum_{n=1}^{\infty}\frac{\overbrace{\cancel{(-1)^n(-1)^n}}^{\overset{\text{Even}}{(-1)^{2n}}\to 1}\cancel{4^n}}{(5n+1)\cdot\cancel{4^n}} = \sum_{n=1}^{\infty}\frac{1}{5n+1} \approx \sum_{n=1}^{\infty}\frac{1}{n}$$

Diverges (Harmonic) p-Series $p=1$

$$\lim_{n\to\infty}\frac{\frac{1}{5n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{\cancel{5n+1}}^{\frac{1}{5}} = \frac{1}{5}$$ Finite, Non-Zero

$\Rightarrow$ Series also Diverges by LCT

Take $x = \frac{3}{5}$. Series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n[\cancel{5}(\frac{3}{\cancel{5}})+1]^n}{(5n+1)\;4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n\cancel{4^n}}{(5n+1)\cancel{4^n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{5n+1}$$

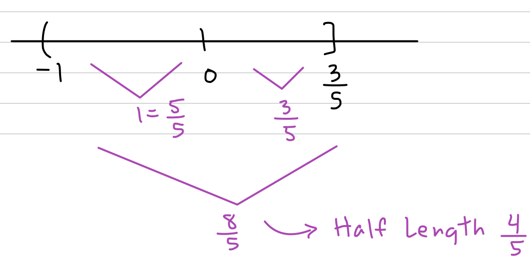
1. Isolate $b_n = \frac{1}{5n+1} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\cancel{\frac{1}{5n+1}}^{\to 0}_{\infty} = 0$
3. Terms Decreasing

$$b_{n+1} = \frac{1}{5(n+1)+1} = \frac{1}{5n+6} \leq \frac{1}{5n+1} = b_n$$

Series Converges by AST

Finally, Interval of Convergence $I = \left(-1, \frac{3}{5}\right]$

Radius of Convergence $R = \frac{4}{5}$



2(a) $\ln\left(1+\frac{x^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^2}{4}\right)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+2}}{4^{n+1}(n+1)}$

Need $\left|\frac{x^2}{4}\right| < 1 \Rightarrow |x|^2 < 4 \Rightarrow |x| < 2 \Rightarrow R = 2$

2(b) $6x^3 \arctan(6x) = 6x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (6x)^{2n+1}}{2n+1} = 6x^3 \sum_{n=0}^{\infty} \frac{(-1)^n \, 6^{2n+1} x^{2n+1}}{2n+1}$

Need $|6x| < 1$

$|x| < \frac{1}{6}$

$-\frac{1}{6} < x < \frac{1}{6}$

$R = \frac{1}{6}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \, 6^{2n+2} x^{2n+4}}{2n+1}$$

$R = \infty$

2(c) $\frac{d}{dx}\left(8x^4 \sin(8x)\right) = \frac{d}{dx}\left(8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (8x)^{2n+1}}{(2n+1)!}\right) = \frac{d}{dx} \, 8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n \, 8^{2n+1} x^{2n+1}}{(2n+1)!}$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n \, 8^{2n+2} x^{2n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \, 8^{2n+2} (2n+5) x^{2n+4}}{(2n+1)!}$$

$R = \infty$ STILL After Differentiation

$R = \infty$

2(d) $\int x^3 e^{-x^4} dx = \int x^3 \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} dx = \int x \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} dx$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{4n+4}}{n! \, (4n+4)} + C$$

$R = \infty$ STILL After Integration

3(a) $\int_0^1 x^2\cos(x^3)\,dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}\,dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}\,dx$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n)!\,(6n+3)}\Bigg|_0^1$$

$$= \underbrace{\frac{x^3}{1\cdot 3}}_{n=0} - \underbrace{\frac{x^9}{2!\cdot 9}}_{n=1} + \underbrace{\frac{x^{15}}{\underset{24}{4!}\cdot(15)}}_{n=2} - \cdots\Bigg|_0^1$$

$$= \underbrace{\frac{1}{3} - \frac{1}{18}}_{\text{Keep}} + \frac{1}{360} - \cdots - (\cancel{0-0+0-\cdots})^{0}$$

$$\approx \frac{1}{3} - \frac{1}{18} = \frac{6}{18} - \frac{1}{18} = \frac{5}{18} \longleftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem (A.S.E.T) we can Estimate the Full Sum using only the first two terms with error at most $\frac{1}{360} < \frac{1}{50}$ as desired

3(b) $\frac{1}{e} = e^{-1} = 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \cdots = \cancel{1} - \cancel{1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \cdots$

$$= \underbrace{\frac{1}{2} - \frac{1}{6} + \frac{1}{24}}_{\text{Keep}} - \frac{1}{120} + \frac{1}{6!} - \cdots$$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\approx \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{12}{24} - \frac{4}{24} + \frac{1}{24} = \frac{9}{24}$$

Estimate

Using the Alternating Series Estimation Theorem (A.S.E.T) we can Estimate the Full Sum using only the first two terms with error at most $\frac{1}{120} < \frac{1}{100}$ as desired

4 a. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n} \cdot \frac{\pi}{3}}{(2n+1)! \cdot \frac{\pi}{3}}$

$9^n = (3^2)^n = 3^{2n}$

$$= \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{3}{\pi} \cancel{\sin\left(\frac{\pi}{3}\right)}^{\frac{\sqrt{3}}{2}} = \frac{3\sqrt{3}}{2\pi}$$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

4b. $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\ldots = -\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\right)$

$$= -\ln(1+1) = -\ln 2$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

4c. $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n+1}}{4!(2n)!} = -\frac{\pi}{24}\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n}}{(2n)!} = -\frac{\pi}{24}\cdot \cancel{\cos\pi}^{-1} = \frac{\pi}{24}$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

4d $$\sum_{n=0}^{\infty}\frac{(-1)^n 2^{n+1}(\ln 3)^n}{5\cdot n!} = \frac{2}{5}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n (\ln 3)^n}{n!} = \frac{2}{5}\sum_{n=0}^{\infty}\frac{(-2\ln 3)^n}{n!}$$

$$= \frac{2}{5}e^{-2\ln 3} = \frac{2}{5}\cancel{e}^{\cancel{\ln}(3^{-2})} = \frac{2}{5}\cdot 3^{-2} = \frac{2}{5}\cdot\frac{1}{9} = \frac{2}{45}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

4e. $$-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\ldots = \left(4\cancel{\arctan 1}^{\frac{\pi}{4}}\right) - 4 = \cancel{4}\cdot\frac{\pi}{\cancel{4}} - 4 = \pi - 4$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

$$4\arctan 1 = 4\left(1-\frac{1}{3}+\frac{1}{5}-\ldots\right) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots$$

4(f) $$\pi^2 - \frac{\pi^4}{3!} + \frac{\pi^6}{5!} - \frac{\pi^8}{7!} + \ldots = \pi\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots\right)$$

$$= \pi\cdot\cancel{\sin\pi}^{0} = 0$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

5. $$\lim_{x\to 0} \frac{1-\cos(2x)}{e^{-x}-1+x} = \lim_{x\to 0} \frac{1-\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots\right)}{1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}+\dots-1+x}$$

$$= \lim_{x\to 0} \frac{\cancel{1}-\cancel{1}+\frac{2^2x^2}{2!}-\frac{2^4x^4}{4!}+\frac{2^6x^6}{6!}-\dots}{\cancel{1}-\cancel{x}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\dots-\cancel{1}+\cancel{x}}$$

Need all ...

$$= \lim_{x\to 0} \frac{\frac{2^2x^2}{2!}-\frac{2^4x^4}{4!}+\frac{2^6x^6}{6!}-\dots}{\frac{x^2}{2}-\frac{x^3}{3!}+\frac{x^4}{4!}-\dots} \quad \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to 0} \frac{\frac{4}{2}-\cancelto{0}{\frac{2^4x^2}{4!}+\frac{2^6x^4}{6!}-\dots}}{\frac{1}{2}-\cancelto{0}{\frac{x}{3!}+\frac{x^2}{4!}-\dots}} = \frac{\cancelto{2}{\frac{4}{2}}}{\frac{1}{2}} = 4$$

☆ Check answer with L'Hôpital's Rule

$$\lim_{x\to 0} \frac{1-\cos(2x)}{e^{-x}-1+x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{2\sin(2x)}{-e^{-x}+1} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{4\cancelto{1}{\cos(2x)}}{\cancel{-}\cancelto{1}{e^{-x}}} = \frac{4}{1} = 4$$ Match!

6. $$\ln(9+x^2) = \int \frac{2x}{9+x^2}\,dx = \int 2x\left(\frac{1}{9+x^2}\right)dx = \int \frac{2x}{9}\left(\frac{1}{1+\frac{x^2}{9}}\right)dx = \int \frac{2x}{9}\left(\frac{1}{1-\left(-\frac{x^2}{9}\right)}\right)dx$$

$$= \int \frac{2x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n dx = \int \frac{2x}{9}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{9^n}\,dx = \int 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{9^{n+1}}\,dx$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(2n+2)} + C = 2\left(\underbrace{\frac{x^2}{9\cdot 2}}_{n=0} - \underbrace{\frac{x^4}{9^2\cdot 4}}_{n=1} + \underbrace{\frac{x^6}{9^3\cdot 6}}_{n=2} - \dots\right) + C$$

Test the center $x=0$ to solve for $+C$

$$\ln\cancelto{\ln 9}{(9+0)} = 2\left(\cancelto{0}{0-0+0-\dots}\right)+C \Rightarrow C=\ln 9$$

Finally, $$\ln(9+x^2) = 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(2n+2)} + \ln 9$$

## Optional Bonus #1

$\sum_{n=0}^{\infty} \frac{n}{3^n}$ looks like $\sum_{n=0}^{\infty} n x^n$ where $x = \frac{1}{3}$

work in reverse to find function

$$\sum_{n=0}^{\infty} n x^n = \sum_{n=0}^{\infty} n \cdot x^{n-1} \cdot x = x \sum_{n=0}^{\infty} n x^{n-1} = x \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$\frac{1}{1-x}$

looks like derivative

$$= x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x\left(\frac{1}{(1-x)^2}\right) = \frac{x}{(1-x)^2}$$

$(1-x)^{-1} = -(1-x)^{-1} \cdot (-1) = (1-x)^{-2}$

Plug in $x = \frac{1}{3}$

$$\sum_{n=0}^{\infty} \frac{n}{3^n} = \sum_{n=0}^{\infty} n\left(\frac{1}{3}\right)^n = \frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} = \frac{\frac{1}{3}}{\frac{4}{9}} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{9}^{\,3}}{4} = \frac{3}{4}$$

$\frac{2}{3}$ $\quad \frac{9}{4}$

## Optional Bonus # 2

$$x^3 \arctan(x^5) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^5)^{2n+1}}{2n+1} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n+5}}{2n+1}$$

$n=0 \qquad n=1 \qquad n=2 \qquad n=3$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n+8}}{2n+1} = \frac{x^8}{1} - \frac{x^{18}}{3} + \frac{x^{28}}{5} - \frac{x^{38}}{7} + \cdots$$

$0x^{27}$

coefficient 0

Match to General MacLaurin Series

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(27)}(0)}{(27)!}x^{27} + \frac{f^{(28)}(0)}{(28)!}x^{28} + \cdots$$

Equate coefficients of like Powered Terms

$$\frac{f^{(27)}(0)}{(27)!} = 0 \Rightarrow f^{(27)}(0) = 0$$

$$\frac{f^{(28)}(0)}{(28)!} = \frac{1}{5} \Rightarrow f^{(28)}(0) = \frac{(28)!}{5}$$