Exam #3 Answer Key Spring 2021

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (6x+1)^n}{(6n+1)^2 \cdot 7^n}$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}(6x+1)^{n+1}}{[6(n+1)+1]^2 \cdot 7^{n+1}}}{\dfrac{\cancel{(-1)^n}(6x+1)^n}{(6n+1)^2\cdot 7^n}} \right| = \lim_{n\to\infty} \underbrace{\left|\frac{(6x+1)^{n+1}}{(6x+1)^n}\right|}_{|6x+1|} \cdot \underbrace{\frac{(6n+1)^2}{(6n+7)^2}}_{\left(\frac{6n+1}{6n+7}\right)^2 \to 1} \cdot \underbrace{\frac{7^n}{7^{n+1}}}_{7}$$

Converges by Ratio Test when

$$= \frac{|6x+1|}{7} < 1$$

$$\Rightarrow |6x+1| < 7 \Rightarrow -7 < 6x+1 < 7 \Rightarrow -8 < 6x < 6$$

$-1 \quad -1 \quad -1$

$-8/6 < x < 1$

$-4/3$

Test Convergence at Endpoints

Take $x = -4/3$. Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n \left[6\left(-4/3\right)+1\right]^n}{(6n+1)^2 \cdot 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-7)^n}{(6n+1)^2 \cdot \cancel{7^n}} = \sum_{n=1}^{\infty} \frac{\overbrace{(-1)^{2n}}^{\text{even},\ 1}}{(6n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(6n+1)^2} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$2 \quad -8$

$(-1)^n 7^n$

Converges, p-Series $p = 2 > 1$

Bound Terms

$$\frac{1}{(6n+1)^2} \le \frac{1}{(6n)^2} = \frac{1}{36n^2} \le \frac{1}{n^2}$$

$\Rightarrow$ Series Converges by CT

[OR LCT works, Limit $= \frac{1}{36}$

$$\lim_{n\to\infty} \frac{\frac{1}{(6n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(6n+1)^2} = \cdots = \frac{1}{36}$$

Finite, Non-Zero]

Take $x = 1$. Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n (6+1)^n}{(6n+1)^2 \cdot \cancel{7^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(6n+1)^2} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{(6n+1)^2}$$

A.S. Converges as shown above by C.T.

$\Rightarrow$ O.S. Converges by ACT [OR AST with $b_n = \frac{1}{(6n+1)^2}$]

Finally, $I = \boxed{[-4/3, 1]}$

$R = \boxed{\frac{7}{6}}$

$-4/3 \quad 0 \quad 1$

$\underbrace{4/3 \quad 1}$

Total: $7/3$ $\to$ Half $7/6$

2. $\sum_{n=1}^{\infty} n^n (x-5)^n$

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}(x-5)^{n+1}}{n^n \quad (x-5)^n} \right| = \lim_{n\to\infty} \underbrace{\frac{(n+1)^n}{n^n}}_{e} \cdot \underbrace{(n+1)}_{\infty} |x-5| = \infty > 1$$ Diverges by R.T.

unless $x=5$ (when $L=0<1$)

$I = \{5\}$

$[R=0]$

OR $\sum_{n=1}^{\infty} n!\,(x-5)^n$

OR $\sum_{n=1}^{\infty} (2n)!\,(x-5)^n$

OR $\sum_{n=1}^{\infty} (3n)!\,(x-5)^n$

$\vdots$

3. $I = (5,7)$ translates to $|x-6|<1$ so choose Geometric Series

$(5 \quad 6 \quad 7)$

center point.

$$\sum_{n=0}^{\infty} (x-6)^n$$

Converges by GST when

$|r| = |x-6| < 1$

$-1 < x-6 < 1$

$+6 \quad +6 \quad +6$

$5 < x < 7$ ✓

Diverges by GST otherwise

$|x-6| \geq 1$ (including endpoints)

4(a) $$\frac{x^2}{4+x} = x^2\left[\frac{1}{4+x}\right] = \frac{x^2}{4}\left[\frac{1}{1+x/4}\right] = \frac{x^2}{4}\left[\frac{1}{1-(-x/4)}\right] = \frac{x^2}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n$$

$$= \frac{x^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{4^{n+1}}$$

To Conv. by GST

Need $\left|-\frac{x}{4}\right| < 1$

$\Rightarrow |x|<4 \Rightarrow R=4$

4(b) $6x^4 \overbrace{\arctan(6x)}^{R=1} = 6x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (6x)^{2n+1}}{2n+1} = 6x^4 \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n+1} x^{2n+1}}{2n+1}$

Conv. if $|6x|<1$

$\Rightarrow |x|<1/6$

$\Rightarrow R=1/6$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n+2} x^{2n+5}}{2n+1}$$

5(a) $\frac{d}{dx}\left(5x^2 e^{-x^3}\right) = \frac{d}{dx}\left(5x^2 \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!}\right) = \frac{d}{dx}\left(5x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}\right)$

$$= \frac{d}{dx}\left(5\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{n!}\right) = 5\sum_{n=0}^{\infty} \frac{(-1)^n (3n+2) x^{3n+1}}{n!}$$

5(b) $\int x^3 \sin(8x^4)\,dx = \int x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (8x^4)^{2n+1}}{(2n+1)!}\,dx$

$$= \int x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+1} x^{8n+4}}{(2n+1)!}\,dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+1} x^{8n+7}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+1} x^{8n+8}}{(2n+1)!\,(8n+8)} + C$$

6. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\cos(1/2) = 1 - \frac{(1/2)^2}{2!} + \frac{(1/2)^4}{4!} - \frac{(1/2)^6}{6!} + \cdots = 1 - \frac{(1/4)}{2} + \frac{\left(\frac{1}{16}\right)}{24} - \frac{\left(\frac{1}{64}\right)}{720} + \cdots$$

$$= \underbrace{1 - \frac{1}{8} + \frac{1}{384}} - \frac{1}{46{,}080} + \cdots \approx 1 - \frac{1}{8} + \frac{1}{384} = \frac{384}{384} - \frac{48}{384} + \frac{1}{384} = \frac{337}{384}$$ ← Estimate

Using ASET, we can estimate the full sum $\cos(1/2)$ using only the first three terms with error at most $|\text{First Neglected Term}| = \frac{1}{46{,}080} < \frac{1}{10{,}000}$ as desired.

7(a) $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^{n-1}(2n+1)!}\cdot\frac{9}{9} = 9\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1}}{\underbrace{9^n}_{3^{2n}}(2n+1)!} = 9\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1}}{3^{2n}(2n+1)!}\cdot\frac{3}{3}$$

$$= 27\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!} = 27\sum_{n=0}^{\infty}\frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = 27\overbrace{\sin\left(\frac{\pi}{3}\right)}^{\frac{\sqrt{3}}{2}} = \frac{27\sqrt{3}}{2}$$

7(b) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{1}{3}\overbrace{\arctan(1)}^{\frac{\pi}{4}} = \frac{\pi}{12}$$

7(c) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\left[(\sqrt{3})^2\right]^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n}}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{\sqrt{3}}\right)^{2n}}{2n+1}\cdot\frac{1/\sqrt{3}}{1/\sqrt{3}} = \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

$$= \sqrt{3}\,\underbrace{\arctan\left(\frac{1}{\sqrt{3}}\right)}_{\pi/6} = \frac{\pi\sqrt{3}}{6} \quad \text{OR} \quad \frac{\pi}{2\sqrt{3}}$$

7(d) $$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots = -\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right)$$

$$= -\ln(1+1) = -\ln 2$$

7(e) $$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n}}{4!\,(2n)!} = \frac{-1}{4!}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!} = \frac{-1}{24}\overbrace{\cos(\pi)}^{-1} = \frac{1}{24}$$

7(f) $$\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\cdots = \overbrace{1-1}^{0}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\cdots$$

$$= e^{-1} = \frac{1}{e}$$

8. Chart Method

(a) $f(x)=\sin x$ $\quad f(0)=\sin 0=0$

$f'(x)=\cos x$ $\quad f'(0)=\cos 0=1$

$f''(x)=-\sin x$ $\quad f''(0)=0$

$f'''(x)=-\cos x$ $\quad f'''(0)=-1$

$f^{(4)}(x)=\sin x$ $\quad f^{(4)}(0)=0$

$\vdots \qquad \vdots$

Maclaurin Series

$$\cancel{f(0)}^{0}+\cancel{f'(0)}^{1}x+\frac{\cancel{f''(0)}^{0}x^2}{2!}+\frac{\cancel{f'''(0)}^{-1}x^3}{3!}+\frac{\cancel{f^{(4)}(0)}^{0}x^4}{4!}+\frac{\cancel{f^{(5)}(0)}^{1}x^5}{5!}+\cdots$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(b) $$\sin x=\frac{d}{dx}\left(-\cos x\right)=\frac{d}{dx}\left[-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots\right)\right]$$

$$=\frac{d}{dx}\left[-1+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\cdots\right]$$

$$=0+\frac{1}{\cancel{2!}}(\cancel{2}x)-\frac{1}{\underset{3!}{\cancel{4!}}}(\cancel{4}x^3)+\frac{1}{\underset{5!}{\cancel{6!}}}(\cancel{6}x^5)-\frac{1}{\underset{7!}{\cancel{8!}}}(\cancel{8}x^7)+\cdots$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(c) $$\sin x=\int\cos x\,dx=\int\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\,dx$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{\underbrace{(2n)!\,(2n+1)}_{(2n+1)!}}+C$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\cancel{C}^{0}$$

Test $x=0$

$\cancel{\sin 0}^{0}=0-0+0-0+\cdots+C \Rightarrow$ Solve $C=0$

Finally, $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

9. $x = (\arctan t) - t \qquad y = 2\sinh^{-1} t$

$$\frac{dx}{dt} = \frac{1}{1+t^2} - 1 \qquad \frac{dy}{dt} = \frac{2}{\sqrt{1+t^2}}$$

$$\text{Arclength} = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2} - 1\right)^2 + \left(\frac{2}{\sqrt{1+t^2}}\right)^2}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2}\right)^2 - \frac{2}{1+t^2} + 1 + \frac{4}{1+t^2}}\, dt = \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2}\right)^2 + \frac{2}{1+t^2} + 1}\, dt$$

$$= \int_0^{\sqrt{3}} \cancel{\sqrt{\left(\frac{1}{1+t^2} + 1\right)^2}}\, dt = \int_0^{\sqrt{3}} \frac{1}{1+t^2} + 1\, dt = \arctan t + t \Big|_0^{\sqrt{3}}$$

$$= \underbrace{\arctan\sqrt{3}}_{\pi/3} + \sqrt{3} - (\underbrace{\arctan 0}_{0} - 0) = \frac{\pi}{3} + \sqrt{3}$$

Optional, For Fun:

$$\lim_{x \to 0} \frac{xe^x - \sin x}{\ln(1+x) - \arctan x} \quad \frac{0}{0} \quad \underset{L'H}{=} \lim_{x \to 0} \frac{xe^x + e^x - \cos x}{\frac{1}{1+x} - \frac{1}{1+x^2}} \quad \frac{0}{0}$$

$$(1+x)^{-1} - (1+x^2)^{-1}$$

$$\underset{L'H}{=} \lim_{x \to 0} \frac{xe^x + e^x + e^x + \sin x}{-(1+x)^{-2} + (1+x^2)^{-2}(2x)}$$

$$= \lim_{x \to 0} \frac{\underbrace{xe^x}_{\to 0} + \underbrace{2e^x}_{\to 2} + \underbrace{\sin x}_{\to 0}}{\underbrace{-\frac{1}{(1+x)^2}}_{\to -1} + \underbrace{\frac{2x}{(1+x^2)^2}}_{\to 0}}$$

$$= -2$$

Redo with Series

$$\lim_{x\to 0} \frac{xe^x - \sin x}{\ln(1+x) - \arctan x}$$

$$= \lim_{x\to 0} \frac{x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)}$$

$$= \lim_{x\to\infty} \frac{\cancel{x} + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots \cancel{-x} + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots}{\cancel{x} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cancel{\frac{x^5}{5}} - \cdots \cancel{-x} + \frac{x^3}{3} \cancel{- \frac{x^5}{5}} + \frac{x^7}{7} - \cdots}$$

$$= \lim_{x\to\infty} \frac{x^2 + \frac{x^3}{2!} + \frac{x^3}{3!} + \frac{x^4}{3!} + \frac{x^5}{4!} - \frac{x^5}{5!} + \cdots}{-\frac{x^2}{2} + 2\left(\frac{x^3}{3}\right) - \frac{x^4}{4} + \cdots} \quad \frac{1/x^2}{1/x^2}$$

[OR Factor out x^2]

$$= \lim_{x\to\infty} \frac{1 + \frac{x}{2!} + \frac{x}{3!} + \frac{x^2}{3!} + \frac{x^3}{4!} - \frac{x^3}{5!} + \cdots}{-\frac{1}{2} + \frac{2x}{3} - \frac{x^2}{4} + \cdots}$$

0 0 0 0 0 0 0

$$= \frac{1}{(-1/2)} = -2$$

Match above!

Note: you can simplify these similar terms but not necessary since they'll approach 0 anyhow