Math 121 Exam 2 Spring 2023 Answer Key

1(a) $\int_0^e x^2\ln(x^2)\,dx = \lim_{t\to 0^+}\int_t^e x^2\ln(x^2)\,dx = \lim_{t\to 0^+} \frac{x^3}{3}\cdot\ln(x^2)\Big|_t^e - \frac{2}{3}\int_t^e x^2\,dx$

$$\boxed{\begin{array}{ll} u=\ln(x^2) & dv = x^2\,dx \\ du = \frac{1}{x^2}(2x)\,dx = \frac{2}{x}\,dx & v=\frac{x^3}{3} \end{array}}$$

$= \lim_{t\to 0^+} \frac{x^3}{3}\cdot\ln(x^2)\Big|_t^e - \frac{2x^3}{9}\Big|_t^e$

$= \lim_{t\to 0^+} \frac{e^3}{3}\cdot\cancel{\ln(e^2)}\,2 - \underbrace{\cancel{\frac{t^3}{3}\ln(t^2)}}_{0\cdot(-\infty)\,\to\,0} - \left(\frac{2e^3}{9} - \cancel{\frac{2t^3}{9}}^{\,0}\right)$

See (*)

$= \frac{2e^3}{3} - \frac{2e^3}{9} = \frac{6e^3}{9} - \frac{2e^3}{9} = \frac{4e^3}{9}$ Converges

Match

(*) $\lim_{t\to 0^+} \underbrace{t^3\cdot\ln(t^2)}_{0\cdot(-\infty)} = \lim_{t\to 0^+}\frac{\ln(t^2)}{\frac{1}{t^3}}\ t^{-3} \overset{L'H}{=} \lim_{t\to 0^+}\frac{\frac{1}{t^2}\cdot(2t)}{\frac{-3}{t^4}}\ -\frac{t^4}{3} = \lim_{t\to 0^+}\frac{-2t^4}{3t} = \lim_{t\to 0^+} -\frac{2\cancel{t^3}^{\,0}}{3} = 0$

Key Note: ln 0 is undefined, so must "sneak attack" 0 using Limit 0^+

1(b) $\int_e^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_e^t (\ln x)\cdot x^{-2}\,dx \overset{IBP}{=} \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_e^t + \int_e^t x^{-2}\,dx$

$$\boxed{\begin{array}{ll} u=\ln x & dv = x^{-2}\,dx \\ du = \frac{1}{x}\,dx & v=\frac{x^{-1}}{-1} = -\frac{1}{x} \end{array}}$$

$= \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_e^t - \frac{1}{x}\Big|_e^t$

$= \lim_{t\to\infty} \underbrace{-\frac{\ln t}{t}}_{\frac{\infty}{\infty}} + \underbrace{\frac{\ln e}{e}}_{\frac{1}{e}} - \cancel{\frac{1}{t}}^{\,0} + \frac{1}{e}$

$\overset{L'H}{=} \lim_{t\to\infty} \frac{-\cancel{\frac{1}{t}}^{\,0}}{1} + \frac{1}{e} + \frac{1}{e} = \frac{2}{e}$ Converges

Match!

1(c) $\displaystyle\int_{-\infty}^{-3} \frac{8-x}{x^2+2x+5}\,dx = \lim_{t\to-\infty}\int_t^{-3}\frac{8-x}{x^2+2x+5}\,dx$ Complete Square $\displaystyle= \lim_{t\to-\infty}\int_t^{-3}\frac{8-x}{(x+1)^2+4}\,dx$

$(x+1)^2 = x^2+2x+1$
$+4 \qquad\qquad +4$

$$= \lim_{t\to-\infty}\int_{t+1}^{-2}\frac{8-(u-1)}{u^2+4}\,du = \lim_{t\to-\infty}\int_{t+1}^{-2}\frac{9-u}{u^2+4}\,du$$

$\boxed{\begin{array}{l} u = x+1 \Rightarrow x = u-1 \\ du = dx \end{array}}$

$$= \lim_{t\to-\infty}\int_{t+1}^{-2}\frac{9}{u^2+4} - \frac{u}{u^2+4}\,du$$

$x = t \Rightarrow u = t+1$
$x = -3 \Rightarrow u = -3+1 = -2$

$$= \lim_{t\to-\infty} \frac{9}{2}\arctan\left(\frac{u}{2}\right) - \frac{1}{2}\ln|u^2+4| \Big|_{t+1}^{-2}$$

$$= \lim_{t\to-\infty} \underbrace{\frac{9}{2}\arctan\left(\underbrace{\frac{-2}{2}}_{-1}\right)}_{-\frac{\pi}{4}\text{ Finite}} - \underbrace{\frac{1}{2}\ln 8}_{\text{Finite}} \boxed{-}\left(\underbrace{\frac{9}{2}\arctan\left(\underbrace{\frac{t+1}{2}}_{-\infty}\right)}_{-\frac{\pi}{2}\text{ Finite}} \boxed{-} \frac{1}{2}\ln|(\underbrace{t+1}_{-\infty})^2+4|\right)$$

$= \infty$ Diverges
Matches!

1(d) $\displaystyle\int_{-4}^{-3}\frac{8-x}{x^2+2x-8}\,dx = \int_{-4}^{-3}\frac{8-x}{(x-2)(x+4)}\,dx = \lim_{t\to-4^+}\int_t^{-3}\frac{8-x}{(x-2)(x+4)}\,dx$

PFD

$$(x-2)\cancel{(x+4)}\left(\frac{8-x}{\cancel{(x-2)(x+4)}} = \frac{A}{x-2} + \frac{B}{x+4}\right)(x-2)(x+4)$$

$$= \lim_{t\to-4^+}\int_t^{-3}\frac{1}{x-2} - \frac{2}{x+4}\,dx$$

$8-x = A(x+4)+B(x-2)$
$= Ax+4A+Bx-2B$
$= (A+B)x+(4A-2B)$

$$= \lim_{t\to-4^+} \ln|x-2| - 2\ln|x+4| \Big|_t^{-3}$$

Justify size!

Conditions

- $A+B=-1 \Rightarrow A=-1-B$
- $4A-2B=8$ $\quad 4(-1-B)-2B=8$
  $-4-4B-2B=8$
  $-6B=12$
  $B=-2$
  $\Downarrow$
  $A=-1-(-2)=1$

$$= \lim_{t\to-4^+} \underbrace{\ln|-5|}_{\ln 5\text{ Finite}} - 2\underbrace{\ln|-1|}_{0} \boxed{-}\left(\underbrace{\ln|\underbrace{t}_{-4}-2|}_{\ln 6\text{ Finite}} \boxed{-} 2\underbrace{\ln|\underbrace{t+4}_{0^+}|}_{-\infty\text{ blows up}}\right)$$

$= -\infty$ Diverges
Match!

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+7}$ $\hookrightarrow$ $\int_1^{\infty} \frac{1}{x^2+4x+7}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4x+7}\,dx$

Complete Square

$\boxed{\begin{array}{l} u = x+2 \\ du = dx \end{array}}$

$\boxed{\begin{array}{l} x=1 \Rightarrow u=3 \\ x=t \Rightarrow u=t+2 \end{array}}$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{(x+2)^2+3}\,dx$$

$$= \lim_{t\to\infty} \int_3^{t+2} \frac{1}{u^2+3}\,du$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t+2}$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \left(\cancelto{\frac{\pi}{2}}{\arctan\left(\cancelto{\infty}{\frac{t+2}{\sqrt{3}}}\right)} - \cancelto{\frac{\pi}{3}}{\arctan\left(\cancelto{\sqrt{3}}{\frac{3}{\sqrt{3}}}\right)}\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}$$

Integral Converges

$\Rightarrow$ $\boxed{\text{Series Converges by Integral Test}}$

3(a) $\sum_{n=1}^{\infty} n^5+n^4+n^3+n^2+n+1$ Diverges by nTDT because

$$\lim_{n\to\infty} \cancelto{\infty}{n^5+n^4+n^3+n^2+n+1} = \infty \neq 0$$

3(b) $\sum_{n=1}^{\infty} \frac{(n+5)^8}{\ln(n+5)}$ Diverges by nTDT because

$$\lim_{n\to\infty} \frac{(n+5)^8}{\ln(n+5)} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{(x+5)^8}{\ln(x+5)} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} \lim_{x\to\infty} \frac{8(x+5)^7}{\frac{1}{x+5}} = \lim_{x\to\infty} 8\cancelto{\infty}{(x+5)^8} = \infty \neq 0$$

X+5

3(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^8}$ $\overset{\text{A.S.}}{\hookrightarrow}$ $\sum_{n=1}^{\infty} \frac{1}{n^8}$ Converges p-Series $p=8>1$

$\Downarrow$

Original Series $\boxed{\text{Converges by the Absolute Convergence Test}}$ (ACT)

OR

1. Pick $b_n = \frac{1}{n^8} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \cancelto{0}{\frac{1}{n^8}} = 0$ ($n^8 \to \infty$)
3. Terms Decreasing

$b_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = b_n$

$\boxed{\text{Original Series Converges by the Alternating Series Test}}$

3 (d) $\sum_{n=1}^{\infty} \frac{\ln 5}{(n+5)^8} + \frac{(-1)^n 8}{5^{2n+1}} \overset{\text{split}}{=} \sum_{n=1}^{\infty} \frac{\ln 5}{(n+5)^8} + \sum_{n=1}^{\infty} \frac{(-1)^n 8}{5^{2n+1}}$

$\ln 5 \sum_{n=1}^{\infty} \frac{1}{(n+5)^8}$ ☆ $+$ $-\frac{8}{5^3} + \frac{8}{5^5} - \frac{8}{5^7} + \dots$

Converges by CT $\qquad$ $r = -\frac{1}{5^2}$

Constant Multiple of Convergent Series is Convergent

Converges by Geometric Series Test with $|r| = \left|-\frac{1}{5^2}\right| = \frac{1}{25} < 1$

☆ $\sum_{n=1}^{\infty} \frac{1}{(n+5)^8} \approx \sum_{n=1}^{\infty} \frac{1}{n^8}$

Convergent p-Series $p = 8 > 1$

Bound Terms

$\frac{1}{(n+5)^8} \le \frac{1}{n^8}$

$\hookrightarrow \sum_{n=1}^{\infty} \frac{1}{(n+5)^8}$ Converges by CT

Original Series Converges because the Sum of Two Convergent Series is Convergent

3(e) $\sum_{n=1}^{\infty} \arctan\left(\frac{n^8 + \sqrt{3}}{\sqrt{3}n^8 + 5}\right)$ Diverges by the n^{th} Term Divergence Test because

$$\lim_{n\to\infty} \arctan\left(\frac{n^8 + \sqrt{3}}{\sqrt{3}n^8 + 5} \cdot \frac{\frac{1}{n^8}}{\frac{1}{n^8}}\right) = \lim_{n\to\infty} \arctan\left(\frac{1 + \frac{\sqrt{3}}{n^8}}{\sqrt{3} + \frac{5}{n^8}}\right) = \frac{\pi}{6} \neq 0$$

($\frac{\sqrt{3}}{n^8} \to 0$, $\frac{5}{n^8} \to 0$, $\to \frac{1}{\sqrt{3}}$, $\arctan \to \frac{\pi}{6}$)

4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5 + n^4 + n^3 + n^2 + n + 1} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{n^5 + n^4 + n^3 + n^2 + n + 1} \approx \sum_{n=1}^{\infty} \frac{1}{n^5}$ Converges p-Series $p = 5 > 1$

Bound Terms

$\frac{1}{n^5 + n^4 + n^3 + n^2 + n + 1} \le \frac{1}{n^5}$

Note: Limit Comparison also Works here

$\Rightarrow$ Absolute Series also Converges by the Comparison Test

Original Series $\overset{\text{ACT}}{\Leftarrow}$ Converges by the Absolute Convergence Test

5(a) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^5 + 5n + 8}{n^8 + 5}\right) \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{n^5 + 5n + 8}{n^8 + 5} \approx \sum_{n=1}^{\infty} \frac{n^5}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges p-Series $p = 3 > 1$

$$\lim_{n\to\infty} \frac{\frac{n^5 + 5n + 8}{n^8 + 5}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^8 + 5n^4 + 8n^3}{n^8 + 5} \cdot \frac{\frac{1}{n^8}}{\frac{1}{n^8}} = \lim_{n\to\infty} \frac{1 + \frac{5}{n^4} + \frac{8}{n^5}}{1 + \frac{5}{n^8}} = 1$$

Finite Non-Zero

$\Rightarrow$ Absolute Series also Converges by the Limit Comparison Test

$\Rightarrow$ Original Series is Absolutely Convergent by Definition

5(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^5 \cdot n^n \cdot n!}{(2n+1)!}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}(n+1)^5(n+1)^{n+1}(n+1)!}{(2(n+1)+1)!}}{\dfrac{\cancel{(-1)^n}\, n^5\, n^n\, n!}{(2n+1)!}}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{n^5}\cdot\frac{\overbrace{(n+1)^{n+1}}^{(n+1)^n(n+1)}}{n^n}\cdot\frac{\overbrace{(n+1)!}^{(n+1)n!}}{n!}\cdot\frac{\cancel{(2n+1)!}}{\underbrace{(2n+3)!}_{\boxed{(2n+3)(2n+2)}\cancel{(2n+1)!}}}$$

$$\left(\frac{n+1}{n}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)^5 \quad \text{or} \quad \left(1+\frac{1}{n}\right)^5 \to 1$$

$$\left(\frac{1+\frac{1}{n}}{1}\right)^5 \to 1$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}\left(\frac{n+1}{2n+3}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{\cancel{n+1}}{\underbrace{2n+2}_{2(\cancel{n+1})}}\right)$$

$$= \lim_{n\to\infty}\frac{e}{2}\left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right) = \frac{e}{4} < 1$$

$\Rightarrow$ the Series Converges Absolutely by the Ratio Test

5(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{5n+8} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty}\frac{1}{5n+8} \approx \sum_{n=1}^{\infty}\frac{1}{n}$ Diverges p-Series $p=1$

$$\lim_{n\to\infty}\frac{\frac{1}{5n+8}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{5n+8}\,\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{5+\frac{8}{n}} = \frac{1}{5}$$ Finite Non-Zero

$\Rightarrow$ Absolute Series also Diverges by the Limit Comparison Test

AST

1. Pick $b_n = \dfrac{1}{5n+8} > 0$
2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{5n+8} = 0$
3. Terms Decreasing

$$b_{n+1} = \frac{1}{\underbrace{5(n+1)+8}_{5n+13}} \le \frac{1}{5n+8} = b_n$$

Original Series Converges by the Alternating Series Test

Original Series is Conditionally Convergent by Definition

other C.C. examples

$$\sum\frac{(-1)^n}{n+7} \quad \sum\frac{(-1)^n}{6n+9} \quad \sum\frac{(-1)^n n}{n^2+8} \quad \sum\frac{(-1)^n}{\sqrt{n}+9} \quad \sum\frac{(-1)^n\sqrt{n}}{n+6} \quad \ldots$$

Best to keep the choice simple for AST work

Bonus! $\left\{ \dfrac{(\ln n)\, 2^n (n!)^2}{n^{2n}\, (3n)!} \right\}$

Consider the Series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n)\, 2^n (n!)^2}{n^{2n}\, (3n)!}$. Run the Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{\ln(n+1)\, 2^{n+1} ((n+1)!)^2}{(n+1)^{2(n+1)} (3(n+1))!}}{\dfrac{\ln n \cdot 2^n \cdot (n!)^2}{n^{2n} \cdot (3n)!}} \right|$$

$$= \lim_{n\to\infty} \frac{\ln(\cancel{n+1})}{\cancel{\ln n}}^{\;1} \cdot \frac{\underbrace{2^{n+1}}_{\cancel{2^n} \cdot 2}}{\cancel{2^n}} \cdot \frac{\underbrace{((n+1)!)^2}_{(\cancel{n+1})^2 \cancel{(n!)^2}}}{\cancel{(n!)^2}} \cdot \frac{n^{2n}}{\underbrace{(n+1)^{2n+2}}_{(n+1)^{2n} (\cancel{n+1})^2}} \cdot \frac{\cancel{(3n)!}}{\underbrace{(3n+3)!}_{\boxed{(3n+3)(3n+2)(3n+1)} \cancel{(3n)!}}}$$

$$= \lim_{n\to\infty} 2 \cdot \left(\frac{n^n}{(n+1)^n} \right)^2 \cdot \frac{1}{(3n+3)(3n+2)(3n+1)} = 0 < 1$$

$\frac{1}{e}$ $\quad \frac{1}{e^2}$ $\quad 0$ $\quad \infty$

$\Rightarrow$ Series Converges (Absolutely)
by the Ratio Test

Since the Related Series Converges, we can Conclude the terms approach 0 that is $\boxed{\displaystyle\lim_{n\to\infty} \frac{(\ln n)\, 2^n (n!)^2}{n^{2n}\, (3n)!} = 0}$, because otherwise if the terms do not approach 0, then the Series would Diverge by the n^{th} Term Divergence Test which Contradicts the Convergence proof above