Exam 1 Spring 23 Answer Key

1(a)

$$\lim_{x\to0}\frac{\overset{0}{\ln(1+5x)}-\overset{0}{5x}}{\underset{0}{\arcsin(3x)}+\underset{1}{e^{-3x}}\underset{-1}{-1}}\quad \frac{0}{0}$$

$$\underset{\text{L'H}}{=}\lim_{x\to0}\frac{\frac{1}{1+5x}\cdot5-5}{\frac{1}{\sqrt{1-(3x)^2}}\cdot3-3e^{-3x}}$$

$$\underset{\text{prep}}{=}\lim_{x\to0}\frac{5(1+5x)^{-1}-5}{3(1-9x^2)^{-1/2}-3e^{-3x}}$$

$$\underset{\text{L'H}}{=}\lim_{x\to0}\frac{-5(1+5x)^{-2}\cdot5}{-\frac{3}{2}(1-9x^2)^{-3/2}(18x)+9e^{-3x}}$$

$$\underset{\text{rewrite}}{=}\lim_{x\to0}\frac{\frac{-25}{\cancel{(1+5x)^2}}\to-25}{\frac{-27x}{\cancel{(1-9x^2)^{3/2}}}\to0+\cancel{9}\to9\ \cancel{e^{-3x}}\to1}$$

$$=-\frac{25}{9}\quad \text{Match}$$

1(b)

$$\lim_{x\to0^+}\overset{0\cdot(-\infty)}{x^3\ln x}=\lim_{x\to0^+}\frac{\ln x}{\frac{1}{x^3}}\ \frac{-\infty}{\infty}\ \underset{\text{L'H}}{=}\lim_{x\to0^+}\frac{\frac{1}{x}}{\frac{-3}{x^4}}\ \frac{-x^4}{3}=\lim_{x\to0^+}\frac{-\cancel{x^3}\to0}{3}=0\quad\text{Match}$$

$x^{-3}\to-3x^{-4}$

1(c)

$$\lim_{x\to\infty}\left(1-\arctan\left(\frac{3}{x^4}\right)\right)^{x^4}=e^{\lim_{x\to\infty}\ln\left(\left(1-\arctan\left(\frac{3}{x^4}\right)\right)^{x^4}\right)}$$

$$=e^{\lim_{x\to\infty}x^4\cdot\ln\left(1-\arctan\left(\frac{3}{x^4}\right)\right)}\quad \infty\cdot0$$

$$=e^{\lim_{x\to\infty}\frac{\ln\left(1-\arctan\left(\frac{3}{x^4}\right)\right)}{\frac{1}{x^4}}}\quad\frac{0}{0}$$

$3x^{-4}$; $\frac{1}{x^4}\to x^{-4}\to-4x^{-5}$

$$=e^{\lim_{x\to\infty}\frac{\cancel{\frac{1}{1-\arctan\left(\frac{3}{x^4}\right)^{0}}}\to1\ \cdot\ \cancel{\frac{-1}{1+\left(\frac{3}{x^4}\right)^{2}_{0}}}\to-1\ \cdot\left(\cancel{-}\cancel{\frac{12}{x^5}}\right)\to3\quad \cancel{-}\frac{\cancel{x^5}}{\cancel{4}}}{\cancel{-}\frac{4}{x^5}}}$$

$$=e^{1\cdot(-1)\cdot3}=e^{-3}\quad\text{Match}$$

$$2.\quad \int_{-3}^{3} \sqrt{9-x^2}\,dx = \int_{x=-3}^{x=3} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta = 9\int_{x=-3}^{x=3} \cos^2\theta\,d\theta$$

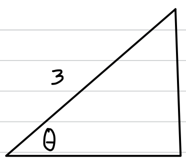
$$\sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

Trig. Sub $\boxed{\begin{array}{l} x = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{array}}$

$\sin\theta = \frac{x}{3}$

$\theta = \arcsin\left(\frac{x}{3}\right)$

$$= 9\int_{x=-3}^{x=3} \frac{1+\cos(2\theta)}{2}\,d\theta = \frac{9}{2}\int_{x=-3}^{x=3} 1+\cos(2\theta)\,d\theta$$



$\Rightarrow \sqrt{9-x^2}$

$$= \frac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Bigg|_{x=-3}^{x=3} = \left(\arcsin\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)\frac{\sqrt{9-x^2}}{3}\right)\Bigg|_{-3}^{3}$$

$\sin(2\theta) = \cancel{2}\sin\theta\cos\theta$

$$= \frac{9}{2}\left(\arcsin\left(\frac{3}{3}\right) + \left(\frac{3}{3}\right)\frac{\sqrt{9-9}}{2} - \left(\arcsin\left(\frac{-3}{3}\right) + \left(\frac{-3}{3}\right)\frac{\sqrt{9-9}}{3}\right)\right)$$

$\frac{3}{3} \to 1$, $\arcsin(1) \to \frac{\pi}{2}$, $\sqrt{9-9} \to 0$, $\frac{-3}{3} \to -1$, $\arcsin(-1) \to -\frac{\pi}{2}$, $\sqrt{9-9} \to 0$

$$= \frac{9}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{9\pi}{2}$$

$\frac{\pi}{2} + \frac{\pi}{2} = \pi$

Match

$$3.\quad \int_0^{\ln\sqrt{3}} \frac{e^{2x}}{3+e^{4x}}\,dx = \int_0^{\ln\sqrt{3}} \frac{e^{2x}}{3+(e^{2x})^2}\,dx = \frac{1}{2}\int_1^3 \frac{1}{3+u^2}\,du = \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_1^3$$

$\boxed{\begin{array}{l} u = e^{2x} \\ du = 2e^{2x}dx \\ \frac{1}{2}du = e^{2x}dx \end{array}}$

$\boxed{\begin{array}{l} x=0 \Rightarrow u = e^0 = 1 \\ x = \ln\sqrt{3} \Rightarrow u = e^{2\ln\sqrt{3}} \\ \quad = e^{\cancel{\ln}(\sqrt{3}^2)} \\ \quad = 3 \end{array}}$

$$= \frac{1}{2\sqrt{3}}\left(\arctan\left(\frac{3}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right)$$

$\frac{3}{\sqrt{3}} \to \sqrt{3}$, $\arctan(\sqrt{3}) \to \frac{\pi}{3}$, $\arctan\left(\frac{1}{\sqrt{3}}\right) \to \frac{\pi}{6}$

$$= \frac{1}{2\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{12\sqrt{3}}$$

$\frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

Match

$$4.\quad \int_1^e x^3\cdot\ln x\,dx = \frac{x^4}{4}\cdot\ln x\Bigg|_1^e - \frac{1}{4}\int_1^e \frac{x^4}{x}\,dx$$

$\frac{x^4}{x} \to x^3$

$\boxed{\begin{array}{ll} u = \ln x & dv = x^3dx \\ du = \frac{1}{x}dx & v = \frac{x^4}{4} \end{array}}$

$$= \frac{x^4}{4}\cdot\ln x\Bigg|_1^e - \frac{x^4}{16}\Bigg|_1^e$$

$$= \frac{e^4}{4}\cdot\ln e - \frac{1}{4}\ln 1 - \left(\frac{e^4}{16} - \frac{1}{16}\right)$$

$\ln e \to 1$, $\ln 1 \to 0$

$$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \frac{3e^4}{16} + \frac{1}{16} = \frac{1+3e^4}{16}$$

$\frac{e^4}{4} = \frac{4e^4}{16}$

Match

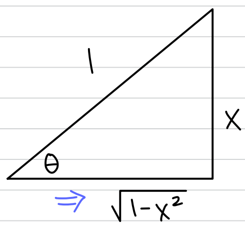
5. $\int x^2 \arcsin x \, dx = \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cdot \cancel{\cos\theta}\, d\theta$

$\sqrt{1-\sin^2\theta} \to \sqrt{\cos^2\theta} \to \cancel{\cos\theta}$

$\boxed{\begin{array}{ll} u=\arcsin x & dv = x^2\,dx \\ du = \frac{1}{\sqrt{1-x^2}}\,dx & v=\frac{x^3}{3}\end{array}}$

$\boxed{\begin{array}{l} x=\sin\theta \\ dx=\cos\theta\, d\theta\end{array}}$

$\theta = \arcsin x$



$\Rightarrow$ $\cos\theta = \sqrt{1-x^2}$

$\boxed{\begin{array}{l} w=\cos\theta \\ dw = -\sin\theta\, d\theta \\ -dw = \sin\theta\, d\theta\end{array}}$

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \sin^3\theta\, d\theta$$ ODD Power

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \sin^2\theta\cdot\sin\theta\, d\theta$$

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int (1-\cos^2\theta)\cdot \sin\theta\, d\theta$$

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\int 1-w^2\, dw$$

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left(w-\frac{w^3}{3}\right) + C$$

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$= \boxed{\frac{x^3}{3}\arcsin x + \frac{1}{3}\left(\sqrt{1-x^2} - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3\right) + C}$$

OR $(1-x^2)^{3/2}$

6. $\displaystyle\int \frac{x}{(x^2+4)^{7/2}}\,dx = \int \frac{x}{\left(\sqrt{x^2+4}\right)^7}\,dx = \int \frac{2\tan\theta}{\left(\sqrt{4\tan^2\theta+4}\right)^{7}} \cdot 2\sec^2\theta\, d\theta$

Don't Drop

$\sqrt{4(\tan^2\theta+1)}$

$\sqrt{4\sec^2\theta}$

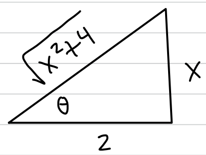
Must Use Trig. Sub here

$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\, d\theta \end{aligned}}$

$\tan\theta = \frac{x}{2}$

$$= \int \frac{2\tan\theta}{\left(\sqrt{4\sec^2\theta}\right)^7} \cdot 2\sec^2\theta\, d\theta$$

$(2\sec\theta)^7$



$\hookrightarrow \cos\theta = \frac{2}{\sqrt{x^2+4}}$

$$= \frac{2^2}{2^7}\int \frac{\tan\theta}{\cancel{\sec^7\theta}}\ \cancel{\sec^2}\theta\, d\theta$$

$\sec^5\theta$

$$= \frac{1}{2^5}\int \tan\theta \cdot \cos^5\theta\, d\theta$$

$\cos^4\theta$

$$= \frac{1}{32}\int \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \cancel{\cos^5}\theta\, d\theta$$

$$= \frac{1}{32}\int \sin\theta \cdot \cos^4\theta\, d\theta \qquad \text{u-sub}$$

$\boxed{\begin{aligned} u &= \cos\theta \\ du &= -\sin\theta\, d\theta \\ -du &= \sin\theta\, d\theta \end{aligned}}$

$$= -\frac{1}{32}\int u^4\, du$$

$$= -\frac{1}{32}\cdot\left(\frac{u^5}{5}\right) + C$$

$$= -\frac{1}{32}\cdot\frac{1}{5}\cdot\left(\frac{2}{\sqrt{x^2+4}}\right)^5 + C$$

$$= -\frac{1}{\cancel{32}}\cdot\frac{1}{5}\cdot\frac{\cancel{32}}{\left(\sqrt{x^2+4}\right)^5} + C$$

$$= \boxed{-\frac{1}{5(x^2+4)^{\frac{5}{2}}} + C} \qquad \text{Match}$$