Limit Comparison Test (LCT)

Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with positive terms. Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ with $0 < C < \infty$. Then

1. If $\sum_{n=1}^{\infty} b_n$ Converges, then $\sum_{n=1}^{\infty} a_n$ Converges.
2. If $\sum_{n=1}^{\infty} b_n$ Diverges, then $\sum_{n=1}^{\infty} a_n$ Diverges.

USED: When your given series behaves more like a simpler series, when n is large, but you may not have a direct, obvious, and helpful bound as with the Comparison Test. Also only used for positive termed series.

USED: LCT does not concern itself with which terms are bigger or smaller. LCT only cares if the given terms and the comparison terms are *about* the same in size, as n gets very big. LCT is the *lazy* version of comparison, and focuses on the idea that the original series and the (simpler?!) comparison series *share the same convergence behavior.*

GOOD FOR: $\sum \frac{\text{polynomial}}{\text{polynomial}}$

NOTE: The order of the stack in the limit of $\frac{a_n}{b_n}$ vs. $\frac{b_n}{a_n}$ is not so important, because you are just trying to decide if C is finite and non-zero. It is often easier (algebraically) to put the given terms in the numerator and the comparison terms in the denominator.

NOTE: If $C = 0$ or $C = \infty$, then you either made an algebraic or miscopy mistake in the Limit computation, OR chose the wrong comparison series to start. Ignore all non-dominant terms as n grows large, and try again.

APPROACH:

- Given the original series, start by ignoring non-dominant terms and decide what the comparison series will be. Again, this Comparison series is usually a p-series or a geometric series.
- Compare the terms. Compute the Limit of the stack of the given terms over the comparison terms that are simpler. Justify the limit answer carefully. No guesses. If you use L'H Rule, you must switch to the related function and the x variable.
- Analyze the comparison series completely, meaning state Declaration of Converge or Diverge, the Convergence Test and the Test conditions checked.
- Make a conclusion about convergence for the original series.

EXAMPLES: Determine and state whether each of the following series **Converges** or **Diverges**. Name any convergence test(s) that you use, and justify all of your work.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7n}{n^9+\sqrt{n}}$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7n}{n^9+\sqrt{n}} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^9} = \sum_{n=1}^{\infty} \frac{1}{n^6}$ which is a Convergent p-series with $p=6>1$.

Next, Check: $\displaystyle\lim_{n\to\infty} \frac{\dfrac{n^3+7n}{n^9+\sqrt{n}}}{\dfrac{1}{n^6}} = \lim_{n\to\infty} \frac{n^9+7n^7}{n^9+\sqrt{n}} \frac{\left(\dfrac{1}{n^9}\right)}{\left(\dfrac{1}{n^9}\right)} = \lim_{n\to\infty} \frac{1+\cancelto{0}{\dfrac{7}{n^2}}}{1+\cancelto{0}{\dfrac{1}{n^{\frac{17}{2}}}}} = 1$ is Finite/Non-zero.

Therefore, the Original Series is also $\boxed{\text{Convergent by Limit Comparison Test}}$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^5+n+6}{3n^6+n^4+5} \approx \sum_{n=1}^{\infty} \frac{n^5}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is a Divergent p-series with $p=1$.

Next Check: $\displaystyle\lim_{n\to\infty} \frac{\dfrac{n^5+n+6}{3n^6+n^4+5}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n^6+n^2+6n}{3n^6+n^4+5} = \lim_{n\to\infty} \frac{1+\cancelto{0}{\dfrac{1}{n^4}}+\cancelto{0}{\dfrac{6}{n^5}}}{3+\cancelto{0}{\dfrac{1}{n^2}}+\cancelto{0}{\dfrac{5}{n^6}}} = \frac{1}{3}$ which is Finite and Non-zero $\left(0<\dfrac{1}{3}<\infty\right)$.

Therefore, these two series share the same behavior, and the Original Series is also $\boxed{\text{Divergent by LCT}}$.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^6+2} \approx \sum_{n=1}^{\infty} \frac{n}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ which is a Convergent p-series with $p=5>1$.

Next Check: $\displaystyle\lim_{n\to\infty} \frac{\dfrac{n}{n^6+2}}{\dfrac{1}{n^5}} = \lim_{n\to\infty} \frac{n^6}{n^6+2} = \lim_{n\to\infty} \frac{1}{1+\cancelto{0}{\dfrac{2}{n^6}}} = 1$ which is Finite and Non-zero.

Therefore, O.S. is also $\boxed{\text{Convergent by LCT}}$. (NOTE: CT would work too)

LAST NOTE: When the Limit C here is Finite and Non-Zero, the test does not directly conclude Converge or Diverge. LCT says if C is Finite and Non-Zero, then the given series has the same convergence or divergence behavior as its comparison series. THINK: *The given series does whatever the comparison series does.*