Absolute Convergence Test

Given a series $\sum_{n=1}^{\infty} a_n$, if the Absolute Series $\sum_{n=1}^{\infty} |a_n|$ Converges, then the Original Series $\sum_{n=1}^{\infty} a_n$ Converges.

USED: When the Absolute Series is easier to analyze.

USED: To avoid analyzing negative signs, or maybe the Alternating Series Test.

BENEFITS: Handling series with positive terms is usually easier. Plus, you have more convergence tests available for positive termed series; think IT, CT, LCT.

WARNING: If you analyze the Absolute Series, and it is NOT convergent, then this convergence test is Inconclusive and not an option for use. It does not immediately follow up with a conclusion about Conditional Convergence without more work. All you know if the the A.S. diverges is that your original series is NOT Absolutely Convergent (A.C.)

NOTE: If the Original Series is already equal to the Absolute series, then Absolute Convergence is the same as Convergence.

NOTE: If the Original Series is not the same as the Absolute Series, then this ACT says that **Absolute Convergence implies Convergence**.

APPROACH:

- Given the original series, find the Absolute Series by ignoring any negative signs.
- Analyze the Absolute Series completely as before. Run all the details of p-Series Test, IT, CT, or LCT. Justify everything. Make a clear conclusion about the convergence of the Absolute Series.
- If the A.S. Converges, then you are done, and the O.S. Converges by ACT.
- If the A.S. diverges, you have **no** conclusion about the O.S. You are stuck running a Convergence test (likely AST) on the O.S.

EXAMPLES: Determine and state whether each of the following series **Converges** or **Diverges**. Name any Convergence test(s) that you use, and justify all of your work.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^5} \rightarrow$ Check the Absolute Series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ which is a Convergent p-Series with $p = 5 > 1$. Then the Original Series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^5}$ is $\boxed{\text{Convergent by ACT}}$.

NOTE: this technique helps you avoid AST here on the O.S.

2. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\sin^2 n}{n^3+5} \rightarrow$ Consider the A.S. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3+5}$.

Bound the terms: $\displaystyle\frac{\sin^2 n}{n^3+5} \leq \frac{\sin^2 n}{n^3} \leq \frac{1}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-Series with $p = 3 > 1$.

Therefore, the A.S. is Convergent by CT. Finally, the O.S. is $\boxed{\text{Convergent by ACT}}$.

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^3+5n}{n^8+7} \rightarrow$ A.S. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+5n}{n^8+7} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ Conv. p-series $p = 5 > 1$.

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n^3+5n}{n^8+7}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{n^8+5n^6}{n^8+7} = \lim_{n\to\infty} \frac{1+\frac{5}{n^2}\nearrow^{0}}{1+\frac{7}{n^8}\nearrow^{0}} = 1$ Finite, Non-zero.

Therefore the A.S. is also convergent by LCT (which by definition means that the A.S. is A.C.). Finally, the O.S. is $\boxed{\text{convergent}}$ by ACT.

4. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\arctan n}{n^2+9} \rightarrow$ A.S. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+9} \approx \sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^2} = \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ Constant Mutliple of a Convergent p-series $p = 2 > 1$ is Convergent.

Bound the terms: $\displaystyle\frac{\arctan n}{n^2+9} \leq \frac{\frac{\pi}{2}}{n^2+9} \leq \frac{\frac{\pi}{2}}{n^2}$. We analyzed the comparison series above.

Finally, the A.S. is convergent by CT which implies that the O.S. is $\boxed{\text{convergent}}$ by ACT.

LAST NOTE: It is very important to finish the full and complete analysis of the Absolute Series. Make a conclusion about the Convergence of the Absolute Series. THEN you can use that to imply convergence of the O.S. using the ACT.