

## Answer Key

- Please see the course webpage for the answer key.

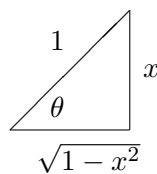
## 1. Compute

$$\begin{aligned}
 \int x^4 \arcsin x \, dx &= \frac{x^5}{5} \arcsin x - \frac{1}{5} \int \frac{x^5}{\sqrt{1-x^2}} \, dx \\
 &= \frac{x^5}{5} \arcsin x - \frac{1}{5} \int \frac{\sin^5 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta \, d\theta \\
 &= \frac{x^5}{5} \arcsin x - \frac{1}{5} \int \frac{\sin^5 \theta}{\sqrt{\cos^2 \theta}} \cos \theta \, d\theta = \frac{x^5}{5} \arcsin x - \frac{1}{5} \int \frac{\sin^5 \theta}{\cos \theta} \cos \theta \, d\theta \\
 &= \frac{x^5}{5} \arcsin x - \frac{1}{5} \int \sin^5 \theta \, d\theta = \frac{x^5}{5} \arcsin x - \frac{1}{5} \int \sin^4 \theta \sin \theta \, d\theta \\
 &= \frac{x^5}{5} \arcsin x - \frac{1}{5} \int (1-\cos^2 \theta)^2 \sin \theta \, d\theta = \frac{x^5}{5} \arcsin x + \frac{1}{5} \int (1-w)^2 \, dw \\
 &= \frac{x^5}{5} \arcsin x + \frac{1}{5} \int 1 - 2w^2 + w^4 \, dw \\
 &= \frac{x^5}{5} \arcsin x + \frac{1}{5} \left( w - \frac{2w^3}{3} + \frac{w^5}{5} \right) \\
 &= \frac{x^5}{5} \arcsin x + \frac{1}{5} \left( \cos \theta - \frac{2\cos^3 \theta}{3} + \frac{\cos^5 \theta}{5} \right) \\
 &= \frac{x^5}{5} \arcsin x + \frac{\sqrt{1-x^2}}{5} - \frac{2(1-x^2)^{\frac{3}{2}}}{15} + \frac{(1-x^2)^{\frac{5}{2}}}{25} + C
 \end{aligned}$$

$u = \arcsin x$	$dv = x^4 dx$
$du = \frac{1}{\sqrt{1-x^2}} dx$	$v = \frac{x^5}{5}$

Trig. Substitute

$x = \sin \theta$
$dx = \cos \theta d\theta$



Substitute

$w = \cos \theta$
$dw = -\sin \theta d\theta$
$-dw = \sin \theta d\theta$

2. Show that  $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(\sin^2 x + 1)^{\frac{7}{2}}} dx = \frac{43}{60\sqrt{2}}$

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{\cos x}{[1 + \sin^2 x]^{\frac{7}{2}}} dx &= \int_0^1 \frac{1}{[1 + u^2]^{\frac{7}{2}}} du \\
 &= \int_{u=0}^{u=1} \frac{1}{[1 + \tan^2 \theta]^{\frac{7}{2}}} \sec^2 \theta d\theta = \int_{u=0}^{u=1} \frac{1}{[\sec^2 \theta]^{\frac{7}{2}}} \sec^2 \theta d\theta \\
 &= \int_{u=0}^{u=1} \frac{1}{(\sqrt{\sec^2 \theta})^7} \sec^2 \theta d\theta = \int_{u=0}^{u=1} \frac{1}{\sec^7 \theta} \sec^2 \theta d\theta \\
 &= \int_{u=0}^{u=1} \frac{1}{\sec^5 \theta} d\theta = \int_{u=0}^{u=1} \cos^5 \theta d\theta = \int_{u=0}^{u=1} \cos^4 \theta \cos \theta d\theta \\
 &= \int_{u=0}^{u=1} (\cos^2 \theta)^2 \cos \theta d\theta = \int_{u=0}^{u=1} (1 - \sin^2 \theta)^2 \cos \theta d\theta \\
 &= \int_{u=0}^{u=1} (1 - w^2)^2 dw = \int_{u=0}^{u=1} 1 - 2w^2 + w^4 dw = w - \frac{2w^3}{3} + \frac{w^5}{5} \Big|_{u=0}^{u=1} \\
 &= \sin \theta - \frac{2 \sin^3 \theta}{3} + \frac{\sin^5 \theta}{5} \Big|_{u=0}^{u=1} = \frac{u}{\sqrt{u^2 + 1}} - \frac{2}{3} \left( \frac{u}{\sqrt{u^2 + 1}} \right)^3 + \frac{1}{5} \left( \frac{u}{\sqrt{u^2 + 1}} \right)^5 \Big|_{u=0}^{u=1} \\
 &= \frac{1}{\sqrt{2}} - \frac{2}{3} \left( \frac{1}{\sqrt{2}} \right)^3 + \frac{1}{5} \left( \frac{1}{\sqrt{2}} \right)^5 - (0 - 0 + 0) \\
 &= \frac{1}{\sqrt{2}} - \frac{2}{3} \left( \frac{1}{2\sqrt{2}} \right) + \frac{1}{5} \left( \frac{1}{4\sqrt{2}} \right) = \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}} \\
 &= \frac{60}{60\sqrt{2}} - \frac{20}{60\sqrt{2}} + \frac{3}{60\sqrt{2}} = \boxed{\frac{43}{60\sqrt{2}}}
 \end{aligned}$$

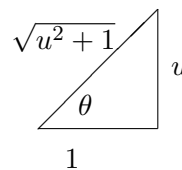
Standard  $u$  substitution to simplify at the start:

$$\begin{array}{l}
 u = \sin x \\
 du = \cos x dx
 \end{array}$$

$$\begin{array}{l}
 x = 0 \Rightarrow u = 0 \\
 x = \frac{\pi}{2} \Rightarrow u = 1
 \end{array}$$

Trig. Substitute

$$\begin{array}{l}
 u = \tan \theta \\
 du = \sec^2 \theta d\theta
 \end{array}$$



Standard  $w$  substitution for *odd* trig. integral  $\int \cos^5 \theta d\theta$  technique:

$$\begin{array}{l}
 w = \sin \theta \\
 dw = \cos \theta d\theta
 \end{array}$$

you can also change one more limit if you rather

$$\dots = \int_0^{\frac{1}{\sqrt{2}}} (1 - w^2)^2 dw = \dots$$

**3.** Compute  $\int \frac{1}{(x^2 + 4)^2} dx =$

$$\int \frac{1}{(x^2 + 4)^2} dx = \int \frac{1}{(4 \tan^2 \theta + 4)^2} 2 \sec^2 \theta d\theta$$

$$= \int \frac{1}{(4 \sec^2 \theta)^2} 2 \sec^2 \theta d\theta = \int \frac{1}{16 \sec^4 \theta} 2 \sec^2 \theta d\theta = \frac{1}{8} \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta$$

$$= \frac{1}{8} \int \frac{1}{\sec^2 \theta} d\theta = \frac{1}{8} \int \cos^2 \theta d\theta = \frac{1}{8} \int \frac{1 + \cos(2\theta)}{2} d\theta$$

$$= \frac{1}{16} \int 1 + \cos(2\theta) d\theta = \frac{1}{16} \left( \theta + \frac{\sin(2\theta)}{2} \right) + C$$

$$= \frac{1}{16} \left( \theta + \frac{2 \sin \theta \cos \theta}{2} \right) + C = \frac{1}{16} (\theta + \sin \theta \cos \theta) + C$$

$$= \frac{1}{16} \left( \arctan \left( \frac{x}{2} \right) + \frac{x}{\sqrt{x^2 + 4}} \left( \frac{2}{\sqrt{x^2 + 4}} \right) \right) + C = \boxed{\frac{1}{16} \left( \arctan \left( \frac{x}{2} \right) + \frac{2x}{x^2 + 4} \right) + C}$$

Trig. Substitute

$x = 2 \tan \theta$ $dx = 2 \sec^2 \theta d\theta$
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