n^{th} Term Divergence Test

Consider a series $\sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} a_n \neq 0$ then the series diverges.

USED: When you suspect the terms of the given series do not approach zero. This is a quick and straightforward test, assuming the limit of the terms is a manageable computation. Understand how the behavior of the terms can determine divergence of the series. Generally, this test is helpful when the series seems a bit "oddball" in form or is not a more natural candidate for another convergence test.

NOTE: This test is **inconclusive** if the terms do approach zero. If $\lim_{n\to\infty} a_n = 0$, then you have more work to do with a **different convergence test**. Simply put, if the terms do approach 0 then you know nothing except that there is *posssible* convergence. Remember, the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series even though the terms $\frac{1}{n}$ shrink to zero as $n \to \infty$.

WARNING: Do **not** create an n^{th} term convergence test. There is no such result or test. Why?

WARNING: Do **not** declare convergence from a divergence test.

APPROACH:

- Given a series, step aside and examine the **terms** of the series.
- If you see that the terms will approach zero as n approaches infinity, then do not waste your time proving that the terms approach zero. This test will not help you. Move onto another appropriate convergence test.
- If you suspect that the terms do **not** go to zero as n approaches infinity, then carefully compute $\lim_{n\to\infty} a_n$. Show the actual limit answer, AND write that it does not equal zero. That is the condition of the test you need to write clearly. Use all of your limit training for sequences. If you need L'H Rule, then make sure that you switch to the related function having x as the variable.
- Think about this diveregnce test in the following way... If $\lim_{n\to\infty} a_n = L \neq 0$, then it says that the *sequence* terms $\{a_n\}_{n=1}^{\infty}$ *converge* to the limit L. However, because the terms do not converge to 0, then the original *series diverges.*

EXAMPLES: Determine and state whether each of the following series **converges** or **diverges**. Name any convergence test(s) that you use, and justify all of your work.

1. $\displaystyle\sum_{n=1}^{\infty} \arctan\left(\frac{\sqrt{3}\, n^2+1}{n^2+\sqrt{3}}\right)$ $\boxed{\text{Diverges}}$ by n^{th} term Divergence Test since

$$\lim_{n\to\infty} \arctan\left(\frac{\sqrt{3}\, n^2+1}{n^2+\sqrt{3}}\right) = \arctan\left(\lim_{n\to\infty}\frac{(\sqrt{3}\, n^2+1)}{(n^2+\sqrt{n})}\cdot\frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}\right)$$

$$= \arctan\left(\lim_{n\to\infty}\frac{\sqrt{3}+\frac{1}{n^2}}{1+\frac{1}{n^{\frac{3}{2}}}}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3} \neq 0$$

2. $\displaystyle\sum_{n=1}^{\infty} n\cdot\arctan\left(\frac{1}{n}\right)$ $\boxed{\text{Diverges}}$ by the n^{th} term Divergence Test since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\cdot\arctan\left(\frac{1}{n}\right) \overset{\infty\cdot 0}{=} \lim_{x\to\infty} x\cdot\arctan\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\arctan\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{x\to\infty}\frac{\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x^2}} = 1 \neq 0$$

3. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+5}\right)^n$ $\boxed{\text{Diverges}}$ by the n^{th} term Divergence Test since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n+5}\right)^n = \lim_{x\to\infty}\left(\frac{x}{x+5}\right)^{x\,(1^\infty)} = \lim_{x\to\infty} e^{\ln\left(\frac{x}{x+5}\right)^x}$$

$$= e^{\lim_{x\to\infty}\ln\left[\left(\frac{x}{x+5}\right)^x\right]} = e^{\lim_{x\to\infty} x\ln\left(\frac{x}{x+5}\right)^{(\infty\cdot 0)}} = e^{\lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+5}\right)}{\frac{1}{x}}^{\frac{\infty}{\infty}}}$$

$$\overset{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{\left(\frac{x+5}{x}\right)\left(\frac{(x+5)(1)-x(1)}{(x+5)^2}\right)}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty}\frac{\left(\frac{x+5}{x}\right)\left(\frac{5}{(x+5)^2}\right)}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x\to\infty}\left(\frac{x+5}{x}\right)\left(\frac{5}{(x+5)^2}\right)(-x^2)} = e^{\lim_{x\to\infty}\left(\frac{-5x}{x+5}\right)^{\frac{\infty}{\infty}}} \overset{\text{L'H}}{=} e^{\frac{-5}{1}} = \frac{1}{e^5} \neq 0$$