Review Packet for Exam #3 Spring 2022

Math 121-D. Benedetto

Interval of Convergence: Find the **interval** and **radius of convergence** for each of the following power series. Analyze convergence at the endpoints carefully, with full justification.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{n^2 4^n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n (x+3)^n}{(n+1)^3 n!}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{n2^n}{n+5}(x+1)^n$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+2)!(x-5)^n}{10^n}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}(2x-1)^n}{4^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^n}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2} x^n$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(3n)!} x^n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^3}(x-1)^n$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^{\frac{1}{2}}}$

13. $\displaystyle\sum_{n=1}^{\infty} nx^n$

14. ~~Challenge: $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$~~

Estimates: Use a Power Series Representation for each of the following functions to **estimate** each one within the given error.

15. Estimate $\cos(1)$ with error less than $\frac{1}{100}$
16. Estimate $e^{-\frac{1}{3}}$ with error less than $\frac{1}{100}$
17. Estimate $\arctan 1$ with error less than .20
18. Estimate $\frac{1}{e}$ with error less than $\frac{1}{10}$
19. Estimate $\sin(1)$ with error less than $\frac{1}{100}$
20. Estimate $\frac{1}{\sqrt{e}}$ with error less than $\frac{1}{100}$
21. Estimate $\sin\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$
22. Estimate $\arctan\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$
23. Estimate $\ln 2$ with error less than $\frac{1}{5}$
24. Estimate $\cos\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$
25. Estimate $\ln\left(\frac{3}{2}\right)$ with error less than $\frac{1}{10}$

MacLaurin Series: Find the MacLaurin Series for each of the following functions, and **state** the corresponding radius of convergence.

26. $f(x) = x^2 e^{-3x^4}$
27. $f(x) = \dfrac{1 - e^{-x}}{x}$
28. $x^4 \ln(1 + x^3)$
29. $\cosh x$
30. $\sinh x$
31. $f(x) = \dfrac{x^6}{1 + 7x}$
32. $f(x) = x \arctan(2x)$

Power Series Representations of Functions: Use a Power Series Representation for each of the following functions to compute the given integral. Estimate each one within the given error.

33. Estimate $\int_0^1 x^2 \cos(x^3)\, dx$ with error less than $\frac{1}{50}$.

34. Estimate $\int_0^{\frac{1}{2}} x \arctan x\, dx$ with error less than 0.01.

35. Estimate $\int_0^1 \sin(x^2)\, dx$ with error less than 0.1.

36. Estimate $\int_0^{\frac{1}{2}} e^{-x^3}\, dx$ with error less than 0.01

Sums: Find the **sum** for each of the following series. (hint: you are allowed to pull an x out of these sums in n. For the harder ones, can you recognize the series as a derivative or integral of another function's power series representation?) Your answer may include x.

37. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}}{3^n}$

38. $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

39. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}$

40. $\sum_{n=0}^{\infty} \frac{(-1)^n 49^n \pi^{2n}}{4^n (2n+1)!}$

41. $\sum_{n=0}^{\infty} \frac{(-9)^n \pi^{2n+1}}{4^n (2n)!}$

42. $\sum_{n=0}^{\infty} \frac{(-\pi^2)^n}{36^n (2n)!}$

43. $\sum_{n=0}^{\infty} \frac{x^{7n+1}}{n!}$

44. $1 - \frac{1}{2} + \frac{1}{2^2\ 2!} - \frac{1}{2^3\ 3!} + \frac{1}{2^4\ 4!} + \dots$

45. $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$

46. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}(n+1)}$

NOTE: Volumes of Revolution Problems #47 − 57 have been cut. Ignore these.

Parametric Equations: Answer each of the following questions, related to the given parametric equations.

58. Let the curve represented by the parametric equations $x = t + \frac{1}{t}$ and $y = 2\ln t$ for $1 \le t \le 3$.

    (a) Find the equation of the tangent line to the curve at the point $(\frac{5}{2}, 2\ln 2)$.

    (b) Find the arclength of this parametric curve for $1 \le t \le 3$.

59. Let the curve represented by the parametric equations $x = \tan t - t$ and $y = \ln(\cos t)$ for $0 \le t \le \frac{\pi}{3}$.

    (a) Find $\dfrac{dy}{dx}$ for the curve when $t = \frac{\pi}{6}$.

    (b) Find the arclength of this parametric curve for $0 \le t \le \frac{\pi}{3}$. (hint: $\sec^2 t - 1 = \tan^2 t$)

60. Let the curve represented by the parametric equations $x = t - e^t$ and $y = 1 - 4e^{\frac{t}{2}}$ for $0 \le t \le \ln 5$.

    (a) Find $\dfrac{dy}{dx}$ for the curve when $t = \ln 4$.

    (b) Find the arclength of this parametric curve for $0 \le t \le \ln 5$.

    ~~(c) Set-up (but do not evaluate) the definite integral representing the surface area of the figure obtained by revolving this curve around the x-axis for $0 \le t \le \ln 5$.~~

61. Let the curve represented by the parametric equations $x = e^t \cos t$ and $y = e^t \sin t$

    for $0 \le t \le \ln \pi$.

    (a) Find the arclength of this parametric curve for $0 \le t \le \ln \pi$.

62. Let the curve represented by the parametric equations $x = 3t^2$ and $y = 2t^3$ for $0 \le t \le \ln 3$.

    (a) Find the equation of the tangent line to the curve at the point $(3, 2)$.

    (b) Find the arclength of this parametric curve for $0 \le t \le 1$.

    ~~(c) Find the surface area obtained by rotating this curve about the y-axis for $0 \le t \le 1$.~~

63. Let the curve represented by the parametric equations $x = \sin^3 t$ and $y = \cos^3 t$ from $t = 0$ to $t = \frac{\pi}{2}$.

    (a) Find the equation of the tangent line to the curve at the point$(\frac{3\sqrt{3}}{8}, \frac{1}{8})$.

    (b) Find the arclength of this parametric curve for $0 \le t \le \dfrac{\pi}{2}$.

    ~~(c) Find the surface area obtained by rotating this curve about the x-axis for $0 \le t \le \dfrac{\pi}{2}$.~~

64. Let the curve represented by the parametric equations $x = 3 - 2t$ and $y = e^t + e^{-t}$.

    (a) Find the arclength of this parametric curve for $0 \le t \le 1$.

    ~~(b) Find the surface area obtained by rotating this curve about the x-axis for $0 \le t \le 1$.~~

    ~~(c) Set-up (but do not evaluate) the definite integral representing the surface area of the figure obtained by revolving this curve around the y-axis for $0 \le t \le 1$.~~

Limits: Compute each of the following limits in two ways: first using L'H Rule and second using series.

65. $\displaystyle\lim_{x\to 0} \frac{\sin(3x) - 3x}{x - \arctan x}$

66. $\displaystyle\lim_{x\to 0} \frac{xe^x - \arctan x}{\ln(1+3x) - 3x}$

Sequence Limits:

67. Use Series to show that $\displaystyle\lim_{n\to\infty} \frac{6^n}{n!} = 0$

68. Use Series to show that $\displaystyle\lim_{n\to\infty} \frac{n^n\ n!}{(3n)!} = 0$

Integrals:

69. Use Series to compute $\displaystyle\int \cos\left(x^2\right) - 1 + \frac{x^4}{2}\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=2}^{\infty}$.

70. Use Series to compute $\displaystyle\int \sin\left(x^2\right) - x^2\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=1}^{\infty}$.

71. Use Series to compute $\displaystyle\int 1 - \cos\left(x^2\right)\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=1}^{\infty}$.

72. Use Series to compute $\displaystyle\int 1 - x^2 - e^{-x^2}\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=2}^{\infty}$.

73. Use Series to compute $\displaystyle\int \arctan(2x) - 2x + \frac{8x^3}{3}\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=2}^{\infty}$.

Derivative Values: ~~**Hint:** Do not compute out the following derivatives manually.~~

~~**Hint:** Write out the definition of the MacLaurin Series for any $f(x)$.~~

74. (a) Write the MacLaurin Series for $\boldsymbol{f(x) = x^5 \sin(x^3)}$. State the Radius of Convergence.

~~(b) Use this series to determine the **eighth** and **ninth** derivatives of $f(x) = x^5 \sin(x^3)$ at $x = 0$. **Simplify** here.~~

75. (a) Write the MacLaurin Series for $\boldsymbol{f(x) = xe^{-x^7}}$. State the Radius of Convergence.

~~(b) Use this series to determine the **twenty first** and **twenty second** derivatives of $f(x) = xe^{-x^7}$ at $x = 0$. Do not simplify here.~~

76. (a) Write the MacLaurin Series for $\boldsymbol{f(x) = x^5 \ln(1 + 3x)}$. State the Radius of Convergence.

~~(b) Use this series to determine the **seventh** and **ninth** derivatives of $f(x) = x^5 \ln(1 + 3x)$ at $x = 0$. Do not simplify here.~~

77. (a) Write the MacLaurin Series for $\boldsymbol{f(x) = x \arctan\left(x^2\right)}$. State the Radius of Convergence.

~~(b) Use this series to determine the **seventh** and **eighth** derivatives of $f(x) = x \arctan\left(x^2\right)$ at $x = 0$. **Simplify** here.~~

78. (a) Find the MacLaurin Series for $\cosh x$.

(b) Demonstrate a **second, different** method/approach from part (a) above, to compute the MacLaurin Series for the same function, $f(x) = \cosh x$.

(c) Demonstrate a **third, different** method/approach from parts (a) and (b) above, to compute the MacLaurin Series for the same function, $f(x) = \cosh x$.

(d) Find the MacLaurin Series for $f(x) = \cosh\left(3x^2\right)$.

~~(e) Use this series to determine the **seventh** and **eighth** derivatives of $f(x) = \cosh\left(3x^2\right)$ at $x = 0$. Do not simplify here.~~

Challenging Sums: Find the sum for each of the following convergent series.

79. ~~$\displaystyle\sum_{n=0}^{\infty} \frac{n}{4^n}$~~

80. ~~$\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{4^n}$~~

81. ~~$\displaystyle\sum_{n=0}^{\infty} \frac{n\,(\ln 3)^n}{n!}$~~

82. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$

83. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1}$