

Exam 2 Spring 2022 Answer Key

$$1(a) \int_{-4}^{-3} \frac{6}{x^2+2x-8} dx = \int_{-4}^{-3} \frac{6}{(x+2)(x-4)} dx = \lim_{t \rightarrow -4^+} \int_t^{-3} \frac{6}{(x+2)(x-4)} dx$$

Given
PFD

$$= \lim_{t \rightarrow -4^+} \int_t^{-3} \frac{1}{x+2} - \frac{1}{x-4} dx = \lim_{t \rightarrow -4^+} \ln|x+2| - \ln|x-4| \Big|_t^{-3}$$

$$= \lim_{t \rightarrow -4^+} \ln|-1| - \ln|-7| - (\ln|t+2| - \ln|t-4|) = -\infty$$

Finite Finite Diverges

$$2(b) \int_{-\infty}^0 \frac{6}{x^2+2x+4} dx = \lim_{t \rightarrow -\infty} \int_t^0 \frac{6}{x^2+2x+4} dx = \lim_{t \rightarrow -\infty} \int_t^0 \frac{6}{(x+1)^2+3} dx$$

Complete
Square

Discriminant
 b^2-4ac
 $= 4-4(1)(4) = -12 < 0$

$u = x+1$
 $du = dx$
 $x = t \Rightarrow u = t+1$
 $x = 0 \Rightarrow u = 1$

$$= \lim_{t \rightarrow -\infty} 6 \int_{t+1}^1 \frac{1}{u^2+3} du$$

$$= \lim_{t \rightarrow -\infty} \frac{6}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) \Big|_{t+1}^1$$

$$= \lim_{t \rightarrow -\infty} \frac{6}{\sqrt{3}} \left(\arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan\left(\frac{t+1}{\sqrt{3}}\right) \right)$$

$$= \frac{6}{\sqrt{3}} \left(\frac{\pi}{6} + \frac{\pi}{2} \right) = \frac{6}{\sqrt{3}} \left(\frac{4\pi}{6} \right) = \frac{4\pi}{\sqrt{3}}$$

Converges

$$1(c) \int_0^1 \ln x dx = \lim_{t \rightarrow 0^+} \int_t^1 \ln x dx = \lim_{t \rightarrow 0^+} x \ln x - x \Big|_t^1$$

$$= \lim_{t \rightarrow 0^+} \ln 1 - 1 - (t \ln t - t) = -1$$

Converges

\times $\lim_{t \rightarrow 0^+} t \cdot \ln t = \lim_{t \rightarrow 0^+} \frac{\ln t}{\frac{1}{t}} \stackrel{\text{L'H}}{=} \lim_{t \rightarrow 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t \rightarrow 0^+} -t = 0$

$$1(d) \int_0^{e^5} \frac{1}{x(25+(\ln x)^2)} dx = \lim_{t \rightarrow 0^+} \int_t^{e^5} \frac{1}{x(25+(\ln x)^2)} dx$$

$$u = \ln x \\ du = \frac{1}{x} dx$$

$$x = t \Rightarrow u = \ln t$$

$$x = e^5 \Rightarrow u = \ln e^5 = 5$$

$$= \lim_{t \rightarrow 0^+} \int_{\ln t}^5 \frac{1}{25+u^2} du$$

$$= \lim_{t \rightarrow 0^+} \frac{1}{5} \arctan\left(\frac{u}{5}\right) \Big|_{\ln t}^5$$

$$= \lim_{t \rightarrow 0^+} \frac{1}{5} \left(\arctan\left(\frac{5}{5}\right) - \arctan\left(\frac{\ln t}{5}\right) \right)$$

$$= \frac{1}{5} \left(\frac{\pi}{4} + \frac{\pi}{2} \right) = \frac{1}{5} \left(\frac{3\pi}{4} \right) = \frac{3\pi}{20}$$

$$2(a) \sum_{n=1}^{\infty} \frac{1}{6^{2n}} + \frac{\ln(2022)}{n^6} = \sum_{n=1}^{\infty} \frac{1}{6^{2n}} + \sum_{n=1}^{\infty} \frac{\ln(2022)}{n^6}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(36)^n} + \underbrace{\ln(2022)}_{\text{Constant}} \sum_{n=1}^{\infty} \frac{1}{n^6}$$

Convergent Geometric Series with

$$|r| = \frac{1}{36} < 1$$

Constant Multiple of Convergent p-Series

$p=6 > 1$ is Convergent

Sum of 2 Convergent Series

is Convergent

$$2(b) \sum_{n=1}^{\infty} \frac{\arctan n}{2022} \text{ Diverges by nTDT b/c}$$

$$\lim_{n \rightarrow \infty} \frac{\arctan n}{2022} = \frac{\frac{\pi}{2}}{4044} \neq 0$$

$$2(c) \sum_{n=1}^{\infty} \frac{n^6}{\ln(n+2022)} \text{ Diverges by nTDT b/c}$$

$$\lim_{n \rightarrow \infty} \frac{n^6}{\ln(n+2022)} = \lim_{x \rightarrow \infty} \frac{x^6}{\ln(x+2022)} \stackrel{\infty/\infty}{=} \lim_{x \rightarrow \infty} \frac{6x^5}{\frac{1}{x+2022}} = \lim_{x \rightarrow \infty} 6x^5(x+2022) = \infty \neq 0$$

$$3. \sum_{n=1}^{\infty} \frac{(-1)^n}{n^6 + 2022} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{n^6 + 2022} \approx \sum_{n=1}^{\infty} \frac{1}{n^6} \text{ Converges p-Series } p=6 > 1$$

Bound Terms

$$\frac{1}{n^6 + 2022} \leq \frac{1}{n^6}$$

\Rightarrow Absolute Series Converges by Comparison Test

\Rightarrow Original Series Converges by Absolute Convergence Test

$$4(a) \sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 6}{n^6 + 2} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{n^2 + 6}{n^6 + 2} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^4} \text{ Converges p-Series } p=4 > 1$$

LCT

$$\lim_{n \rightarrow \infty} \frac{\frac{n^2 + 6}{n^6 + 2}}{\frac{1}{n^4}} = \lim_{n \rightarrow \infty} \frac{n^2 + 6}{n^6 + 2} \cdot \frac{n^4}{1} = \lim_{n \rightarrow \infty} \frac{n^6 + 6n^4}{n^6 + 2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{6}{n^2}}{1 + \frac{2}{n^6}} = 1 \text{ Finite Non-Zero}$$

\Rightarrow Absolute Series Converges by LCT

\Rightarrow Original Series Absolutely Convergent by Definition

$$4(b) \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 6^n \cdot n!}{n^6 \cdot n^n}$$

Ratio Test

$$L = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 6^{n+1} (n+1)!}{(n+1)^6 (n+1)^{n+1}} \cdot \frac{n^6 \cdot n^n}{(-1)^n 6^n n!} \right| = \lim_{n \rightarrow \infty} \frac{6^{n+1} \cdot 6 \cdot (n+1)!}{6^n \cdot (n+1)!} \cdot \frac{n^6}{(n+1)^6} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n \rightarrow \infty} 6 \cdot \left(\frac{n}{n+1} \right)^6 \cdot \frac{n^n}{(n+1)^n} \cdot \frac{1}{n+1}$$

$$= \lim_{n \rightarrow \infty} 6 \cdot \left(\frac{1}{1 + \frac{1}{n}} \right)^6 \cdot \frac{1}{e} = \frac{6}{e} > 1 \text{ Diverges by Ratio Test}$$

4(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+2022}$ $\xrightarrow{\text{A.S.}}$ $\sum_{n=1}^{\infty} \frac{1}{6n+2022} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges p-Series (Harmonic) $p=1$

1. $b_n = \frac{1}{6n+2022} > 0$

2. $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{6n+2022} = 0$

3. Terms Decreasing

$b_{n+1} = \frac{1}{6(n+1)+2022} \leq \frac{1}{6n+2022} = b_n$

Original Series Converges by AST

LCT $\lim_{n \rightarrow \infty} \frac{\frac{1}{6n+2022}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{6n+2022}$

$= \lim_{n \rightarrow \infty} \frac{1}{6 + \frac{2022}{n}} = \frac{1}{6}$ Finite Non-Zero

\Rightarrow Absolute Series Diverges by LCT

Original Series Conditionally Convergent by Definition

Optional Bonus

$\left\{ \frac{(\ln n) \cdot 2^n \cdot (n!)^2}{n^{2n} \cdot (3n)!} \right\}$

First, Consider $\sum_{n=1}^{\infty} \frac{(\ln n) \cdot 2^n \cdot (n!)^2}{n^{2n} \cdot (3n)!}$ which Converges by Ratio Test (see below)

and therefore $\lim_{n \rightarrow \infty} \frac{(\ln n) \cdot 2^n \cdot (n!)^2}{n^{2n} \cdot (3n)!} = 0$ and the sequence converges because

otherwise if the terms did not approach 0, then the series would diverge by nTDT, which would contradict the convergence of the series using Ratio Test

Ratio Test

$\lim_{n \rightarrow \infty} \frac{\frac{(\ln(n+1)) \cdot 2^{n+1} \cdot ((n+1)!)^2}{(n+1)^{2(n+1)} \cdot (3(n+1))!}}{\frac{(\ln n) \cdot 2^n \cdot (n!)^2}{n^{2n} \cdot (3n)!}} = \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln n} \cdot \frac{2^{n+1}}{2^n} \cdot \frac{((n+1)!)^2}{(n!)^2} \cdot \frac{n^{2n}}{(n+1)^{2n+2}} \cdot \frac{(3n)!}{(3n+3)!}$

$= \lim_{n \rightarrow \infty} 2 \left(\frac{n}{n+1} \right)^2 \cdot \frac{1}{(3n+3)(3n+2)(3n+1)} = 0 < 1$ AC. by R.T.

$$(*) \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln n} = \lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\ln x} \stackrel{\frac{\infty}{\infty}}{\text{L'H}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{x}{x+1} \stackrel{\frac{\infty}{\infty}}{\text{L'H}} = \lim_{x \rightarrow \infty} \frac{1}{1} = 1$$