Exam #1 Answer Key Spring 2022

1(a)

$$\lim_{x\to 0} \frac{\ln(1-5x) + \arcsin(5x)}{3xe^x - \arctan(3x)} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{-5}{1-5x} + \frac{5}{\sqrt{1-(5x)^2}}}{3xe^x + 3e^x - \frac{3}{1+(3x)^2}}$$

$\frac{0}{0}$

prep $-5(1-5x)^{-1}$ prep $5(1-25x^2)^{-1/2}$

prep $-3(1+9x^2)^{-1}$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{5(1-5x)^{-2}(-5) - \frac{5}{\cancel{2}}(1-25x^2)^{-3/2}(\cancel{-50x})}{3xe^x + 3e^x + 3e^x + 3(1+9x^2)^{-2}(18x)} = \lim_{x\to 0} \frac{\frac{-25}{(1-5x)^2} + \frac{125x}{(1-25x^2)^{3/2}}}{3xe^x + 6e^x + \frac{54x}{(1+9x^2)^2}} = -\frac{25}{6}$$

Match!

(b)

$0\cdot(-\infty)$

$$\lim_{x\to 0^+} x^5 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^5} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-5}{x^6}} = \lim_{x\to 0^+} \frac{1}{x}\cdot\left(\frac{-x^6}{5}\right) = \lim_{x\to 0^+} \frac{-x^5}{5} = 0$$

$\frac{-\infty}{\infty}$

prep $x^{-5} \to -5x^{-6}$

Flip to simplify

Match!

(c)

1^∞

$$\lim_{x\to\infty} \left(1 - \arcsin\left(\frac{2}{x^4}\right)\right)^{x^4} = e^{\lim_{x\to\infty} \ln\left[\left(1-\arcsin\left(\frac{2}{x^4}\right)\right)^{x^4}\right]}$$

Key: Show All Steps here

$\infty \cdot 0$

$$= e^{\lim_{x\to\infty} x^4 \ln\left(1-\arcsin\left(\frac{2}{x^4}\right)\right)}$$

Flip

$\frac{0}{0}$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1-\arcsin\left(2/x^4\right)\right)}{\frac{1}{x^4}}}$$

$\frac{1}{x^4} \to x^{-4}$

$$\overset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{1-\arcsin(2/x^4)}\cdot\left(-\frac{1}{\sqrt{1-(2/x^4)^2}}\right)\cdot\frac{-8}{x^5}}{\frac{-4}{x^5}}}$$

what cancels?

$$= e^{-2}$$

Match

2. $$\int_e^{e^3} \frac{1}{x\left[3+(\ln x)^2\right]}\,dx = \int_1^3 \frac{1}{3+u^2}\,du = \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_1^3$$

$\boxed{\begin{array}{l} u=\ln x \\ du=\frac{1}{x}\,dx \end{array}}$ $\boxed{\begin{array}{l} x=e \Rightarrow u=\ln e=1 \\ x=e^3 \Rightarrow u=\ln e^3=3 \end{array}}$

$$= \frac{1}{\sqrt{3}}\left(\underbrace{\arctan\left(\underbrace{\frac{3}{\sqrt{3}}}_{\sqrt{3}}\right)}_{\pi/3} - \underbrace{\arctan\left(\frac{1}{\sqrt{3}}\right)}_{\pi/6}\right)$$

$$= \frac{1}{\sqrt{3}}\left(\underbrace{\pi/3}_{2\pi/6} - \pi/6\right) = \frac{1}{\sqrt{3}}\left(\pi/6\right) = \frac{\pi}{6\sqrt{3}}$$

3. $$\int \frac{1}{(4-x^2)^{3/2}}\,dx = \int \frac{1}{\left(\sqrt{4-x^2}\right)^3}\,dx = \int \frac{1}{\left(\sqrt{4-4\sin^2\theta}\right)^3}\cdot 2\cos\theta\,d\theta$$

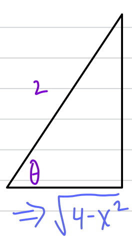
Trig. Sub

$\boxed{\begin{array}{l} x=2\sin\theta \\ dx=2\cos\theta\,d\theta \end{array}}$ $\longrightarrow \sin\theta=\frac{x}{2}$

$$\left(\sqrt{4(1-\sin^2\theta)}\right)^3$$

$$\left(\sqrt{4\cos^2\theta}\right)^3$$

$$(2\cos\theta)^3$$



$$= \int \frac{1}{\cancel{2^3}^{\,2}\cos^{\cancel{3}\,2}\theta}\cdot\cancel{2}\,\cancel{\cos}\theta\,d\theta = \frac{1}{2^2}\int \frac{1}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C = \frac{1}{4}\left(\frac{x}{\sqrt{4-x^2}}\right) + C$$

4. $$\int x^2\arcsin x\,dx = \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

Trig. Sub

IBP

$\boxed{\begin{array}{ll} u=\arcsin x & dv=x^2dx \\ du=\frac{1}{\sqrt{1-x^2}}dx & v=\frac{x^3}{3} \end{array}}$

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cdot\cancel{\cos\theta}\,d\theta$$

$$\sqrt{\cos^2\theta}$$

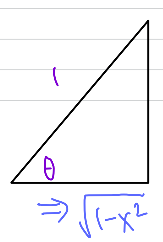
$$\cancel{\cos\theta}$$

Trig. Sub

$\boxed{\begin{array}{l} x=\sin\theta \\ dx=\cos\theta\,d\theta \end{array}}$

ODD Power

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \sin^3\theta\,d\theta$$



$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \underbrace{\sin^2\theta}_{(1-\cos^2\theta)}\cdot\sin\theta\,d\theta$$

"Isolate"

u-sub

$$\boxed{\begin{aligned} u &= \cos\theta \\ du &= -\sin\theta\, d\theta \\ -du &= \sin\theta\, d\theta \end{aligned}}$$

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int (1-\cos^2\theta)\sin\theta\, d\theta$$

"Convert"

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\int 1-u^2\, du$$

"Finish with u-sub"

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left(u - \frac{u^3}{3}\right) + C$$

undo u-sub

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left(\cos\theta - \frac{\cos^3\theta}{3}\right) + C$$

undo Trig Sub

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left(\sqrt{1-x^2} - \frac{\left(\sqrt{1-x^2}\right)^3}{3}\right) + C$$

or $(1-x^2)^{3/2}$

5. $$\int_1^{e^2} \sqrt{x}\ln x\, dx = \frac{2}{3}x^{3/2}\ln x\Big|_1^{e^2} - \frac{2}{3}\int_1^{e^2} \frac{x^{3/2}}{\cancel{x}}\, dx$$

$x^{1/2}$

IBP

$$\boxed{\begin{aligned} u &= \ln x & dv &= \sqrt{x}\, dx \;\; (x^{1/2}) \\ du &= \frac{1}{x}dx & v &= \frac{2}{3}x^{3/2} \end{aligned}}$$

$$= \frac{2}{3}x^{3/2}\ln x\Big|_1^{e^2} - \frac{2}{3}\left(\frac{2}{3}x^{3/2}\right)\Big|_1^{e^2}$$

$$= \frac{2}{3}x^{3/2}\ln x\Big|_1^{e^2} - \frac{4}{9}x^{3/2}\Big|_1^{e^2}$$

$$= \frac{2}{3}(e^2)^{3/2}\cdot \ln(e^2) - \frac{2}{3}\ln 1 - \left(\frac{4}{9}(e^2)^{3/2} - \frac{4}{9}\right)$$

0

$$= \frac{2}{3}e^3\cdot(2) - \frac{4}{9}e^3 + \frac{4}{9}$$

$$= \frac{4}{3}e^3 - \frac{4}{9}e^3 + \frac{4}{9} = \frac{8}{9}e^3 + \frac{4}{9} = \frac{8e^3+4}{9}$$

$\frac{12}{9}$

Match!

6. $$\int \frac{1}{(x^2+4)^2}\, dx = \int \frac{1}{(4\tan^2\theta+4)^2}\cdot 2\sec^2\theta\, d\theta = \int \frac{1}{(4(\tan^2\theta+1))^2}\cdot 2\sec^2\theta\, d\theta$$

$(4\sec^2\theta)^2$

Trig Sub

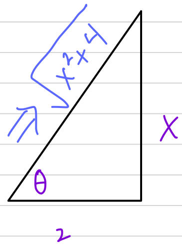
$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\, d\theta \end{aligned}}$$

$\tan\theta = \frac{x}{2} \Rightarrow \theta = \arctan\left(\frac{x}{2}\right)$

$$= \int \frac{1}{\cancel{16}\sec^{\cancel{4}}\theta}\cdot \cancel{2}\sec^2\theta\, d\theta = \frac{1}{8}\int \frac{1}{\sec^2\theta}\, d\theta$$

Flip to Cosine

2 copies left in denominator



$$= \frac{1}{8}\int \cos^2\theta \, d\theta = \frac{1}{8}\int \frac{1+\cos(2\theta)}{2}\, d\theta$$

Half-Angle Identity

$$= \frac{1}{16}\int 1 + \cos(2\theta)\, d\theta = \frac{1}{16}\left(\theta + \frac{\overbrace{\sin(2\theta)}^{\cancel{2}\sin\theta\cos\theta}}{\cancel{2}}\right) + C$$

Double Angle Identity

$$= \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \left(\frac{x}{\sqrt{x^2+4}}\right)\left(\frac{2}{\sqrt{x^2+4}}\right)\right] + C$$

$$= \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right] + C$$