Power Series

We will review the relevant Definitions, Calculus, Strategies, Examples, and Applications.

Definition: A **Power Series Centered at** a has the following form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

where a is a constant value and the Coefficients c_n may depend on n. Here x is the input variable. Note: Convergent for $x = a$.

Definition: A **Power Series Centered at** $a = 0$ has the following form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

where the Coefficients c_n may depend on n. Here x is the input variable. Note: Convergent for $x = 0$.

Think: Power Series are *Infinite Versions of Polynomials*. We will study their Properties and Applications.

First, we can think of Power Series as functions and plug in a specific finite Real x-value. Then the Series becomes an Infinite Series, a sum of infinitely many real numbers.

For Example, $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ is a Power Series Centered at 0.

When we plug in $x = 2$ the Power Series becomes an Infinite Series $\displaystyle\sum_{n=0}^{\infty} 2^n$ which is a Divergent Geometric Series $|r| = 2 > 1$. Hence $x = 2$ is **not** in the Domain.

However, when we plug in $x = \dfrac{1}{4}$ the Power Series becomes an Infinite Series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$ which is a Convergent Geometric Series $|r| = \dfrac{1}{4} < 1$. Hence, $x = \dfrac{1}{4}$ is **in** the Domain.

Question: What is the **Domain** of these function-like Power Series? That is, what is the collection of finite Real numbers that can be plugged into the Power Series and keep it finite, meaning *Convergent* as a series.

Definition: The **Interval of Convergence** is the collection of finite Real numbers at which the Power Series is a finite Infinite Series. That is, the Domain of the Power Series.

Definition: The **Radius of Convergence** is half the length of the full Interval of Convergence. This essentially captures the *size* of the Domain Interval.

Domains

We have a full classification of the Domain of a Power Series $\sum_{n=0}^{\infty} c_n(x-a)^n$. One of the following **Three** things must happen:

1. The Power Series converges only at the Center point $x = a$.

$$I = \{a\}$$
$$R = 0$$

2. The Power Series converges for **all** Real numbers

$$I = (-\infty, \infty)$$
$$R = \infty$$

3. The Power Series converges on a finite Interval Centered at the Center Point a and possibly one or both of the endpoints.

$$I = (a-R, a+r) \text{ or } I = (a-R, a+r] \text{ or } I = [a-R, a+r) \text{ or } I = [a-R, a+r]$$
$$\text{Radius} = R$$

To find the Domain of a Power Series we have two main options:

1. If the Series is Geometric, use the Geometric Series Test. This is beneficial since the Geometric Series Test automatically gives Divergence at the endpoints of the Domain Interval. You do **not** need to test endpoints for the Geometric Examples.

2. Run the Ratio Test. Here you **must** manually check convergence at the endpoints for the finite interval case, since the Ratio Test is Inconclusive at the endpoints, that is, where $L = 1$.

Example: Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=0}^{\infty} \frac{(x-6)^n}{7^n}$$

The Geometric Series Test yields Convergence for $\left|\frac{x-6}{7}\right| < 1$ and Divergence otherwise.

Here we need $|x-6| < 7 \Rightarrow -7 < x-6 < 7 \Rightarrow -1 < x < 13$ and finally, the Interval of Convergence is given by $I = (-1, 13)$ and the Radius is given by $R = 7$. Note that the Center of the Domain Interval is indeed the Center of the Power Series, $x = 6$ here. Cool!

We will now present an example of each of the three Domain options, using the Ratio Test.

Example (Finite Interval): Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ (9x-4)^n}{(n+1) \ 5^n}$$

Use Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(9x-4)^{n+1}}{(n+2) \ 5^{n+1}}}{\dfrac{(-1)^n(9x-4)^n}{(n+1) \ 5^n}} \right| = \lim_{n\to\infty} \left| \frac{(9x-4)^{n+1}}{(9x-4)^n} \right| \cdot \left(\cancelto{1}{\frac{n+1}{n+2}} \right) \cdot \frac{5^n}{5^{n+1}}$$

$$= \frac{|9x-4|}{5}$$

The Ratio Test gives convergence for x when $\dfrac{|9x-4|}{5} < 1$ or $|9x-4| < 5$.

That is $-5 < 9x - 4 < 5 \Longrightarrow -1 < 9x < 9 \Longrightarrow -\dfrac{1}{9} < x < 1$

Manually test Convergence (where $L = 1$) at the Endpoints:

- Take $x = 1$. The original series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \, (9(1)-4)^n}{(n+1) \ 5^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{(n+1) \ 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$ which is convergent by AST:

1. $b_n = \dfrac{1}{n+1} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$

3. $b_{n+1} = \dfrac{1}{n+2} < \dfrac{1}{n+1} = b_n$ or becau se $f(x) = \dfrac{1}{x+1}$ has derivative $f'(x) = -\dfrac{1}{(x+1)^2} <$ 0 so the terms are decreasing.

** Turn the page for the other endpoint check **

Other endpoint:

- Take $x = -\dfrac{1}{9}$. The original series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \left(9\left(-\frac{1}{9}\right) - 4\right)^n}{(n+1)\ 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-5)^n}{(n+1)\ 5^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n 5^n}{(n+1)5^n} = \sum_{n=1}^{\infty} \frac{\overset{\overset{1}{\text{even}}}{\cancel{(-1)^{2n}}}}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n} \text{ Div. harmonic } p\text{-series } p = 1.$$

LCT: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$ which is *finite* and *non-zero.*

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ is also divergent by LCT.

Finally, Interval of Convergence $\boxed{I = \left(-\frac{1}{9}, 1\right]}$ with Radius of Convergence $\boxed{R = \frac{5}{9}}$.

Example (Infinite Interval): Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n\ x^n}{(2n+1)!}$$

Use Ratio Test.

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{(2(n+1)!}}{\frac{(-1)^n x^n}{(2n)!}} \right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| \cdot \left(\frac{(2n)!}{(2n+2)!}\right)$$

$$= \lim_{n\to\infty} \frac{\overset{0}{|x|}}{\underset{\infty}{\cancel{(2n+2)(2n+1)}}} = \frac{0}{\infty} 0 < 1$$

So the Series **Converges by the Ratio Test for all Real numbers x.**

Finally, the Interval of Convergence is $\boxed{I = (-\infty, \infty)}$ and the Radius is $\boxed{R = \infty}$.

Example (Collapsed to a (Center) point Interval): Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty}(3n)!(x-6)^n$$

Use Ratio Test.

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(3(n+1))!(x-6)^{n+1}}{(3n)!(x-6)^n}\right|=\lim_{n\to\infty}\left|\frac{(x-6)^{n+1}}{(x-6)^n}\right|\cdot\left(\frac{(3n+3)!}{(3n)!}\right)$$

$$=\lim_{n\to\infty}|x-6|\cancelto{\infty}{(3n+3)}\cancelto{\infty}{(3n+2)}\cancelto{\infty}{(3n+1)}=\infty>1$$

So the Series **Diverges by the Ratio Test for all Real numbers x UNLESS**

$x-6=0 \Rightarrow x=6$ (when $L=0<1$ which yields convergence by the Ratio Test).

Finally, the Interval of Convergence is $\boxed{I=\{6\}}$ and the Radius is $\boxed{R=0}$.

Term-by-Term Differentation and Integration of Power Series

Next, we can Differentiate or Integrate Power Series *Term-by-Term*. That is, given a Power Series $\displaystyle\sum_{n=0}^{\infty}c_nx^n$ with Radius of Convergence $R>0$ then we can compute its Derivative or Integral and the Radius remains unchanged. (The convergence may change at the endpoints.)

$$\frac{d}{dx}\sum_{n=0}^{\infty}c_nx^n=\sum_{n=0}^{\infty}c_nnx^{n-1} \quad \text{and} \quad \int\sum_{n=0}^{\infty}c_nx^n\,dx=\sum_{n=0}^{\infty}c_n\frac{x^{n+1}}{n+1}+C$$

Example:

$$\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n\ x^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n\ n\ x^{n-1}}{n!}$$

Example:

$$\int\sum_{n=0}^{\infty}\frac{(-1)^n\ x^{4n+2}}{(2n)!}\ dx=\sum_{n=0}^{\infty}\frac{(-1)^n\ x^{4n+3}}{(2n)!\ (4n+3)}+C$$

Power Series Representations for Functions

Question: Does a function have a Power Series Representation?

We start with the Geometric Series form:

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots} \quad \text{valid for } |x| < 1 \text{ by the Geometric Series Test}$$

Next, we can extend this formula to find new Power Series for other functions:

Example: $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \displaystyle\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 + \dots$

valid when $|-x| = |x| < 1$ using the Geometric Series Test. Here the Radius is $R = 1$.

Example: $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = \displaystyle\sum_{n=0}^{\infty}\left(-x^2\right)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 + \dots$

valid when $|-x^2| = |x^2| < 1$ or $|x| < 1$ using the Geometric Series Test. Here the Radius is $R = 1$.

Example: $\dfrac{1}{1+5x} = \dfrac{1}{1-(-5x)} = \displaystyle\sum_{n=0}^{\infty}(-5x)^n = \sum_{n=0}^{\infty}(-1)^n 5^n x^n = 1 - 5x + 5^2x^2 - 5^3x^3 + \dots$

valid when $|-5x| < 1$ or $|x| < \dfrac{1}{5}$ using the Geometric Series Test. Here the Radius is $R = \dfrac{1}{5}$.

Example: $\dfrac{x^3}{6+x} = x^3\left(\dfrac{1}{6-(-x)}\right) = \dfrac{x^3}{6}\left(\dfrac{1}{1-\left(-\frac{x}{6}\right)}\right) = \dfrac{x^3}{6}\left(\displaystyle\sum_{n=0}^{\infty}\left(-\frac{x}{6}\right)^n\right)$

$= \dfrac{x^3}{6}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{6^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{6^{n+1}}$

valid when $\left|-\dfrac{x}{6}\right| < 1$ or $|x| < 6$ using the Geometric Series Test. Here the Radius is $R = 6$.

More Power Series Representations for functions using Intergation

Next, we explored the Power Series for $\ln(1+x)$ and $\arctan x$. We derived these Series using Integration of a known Power Series (derived from the Geometric Series Formula)

$$\ln(1+x) = \int \overset{\text{P.S.}}{\cancel{\frac{1}{1+x}}}\, dx = \int \frac{1}{1-(-x)}\, dx = \int \sum_{n=0}^{\infty} (-x)^n \, dx = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + C$$

Expanding the Power Series a few terms and plugging in the Center point value $x = 0$ we can show that $C = 0$ here

$0 = \ln(1+0) = 0 - 0 + 0 - 0 + \ldots + C \to C = 0$

Finally, $\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}$

Next, $$\arctan x = \int \overset{\text{P.S.}}{\cancel{\frac{1}{1+x^2}}}\, dx = \int \frac{1}{1-(-x^2)}\, dx = \int \sum_{n=0}^{\infty} \left(-x^2\right)^n \, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + C$$

Expanding the Power Series a few terms and plugging in the Center point value $x = 0$ we can show that $C = 0$ here

$0 = \arctan 0 = 0 - 0 + 0 - 0 + \ldots + C \to C = 0$

Finally, $\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$

Taylor and MacLaurin Series

We've been able to find Power Series Representations for a restricted class of functions, related to Geometric Series. Now we want to try and generalize.

Definition: The **Taylor Series for a function f at a** is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

Definition: For the special case when $a = 0$, the Taylor Series centered at $a = 0$ is called the **MacLaurin Series for a function f at** 0. It is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

We used this Definition (or *Chart Method*) to gain Power Series for $\sin x$ and e^x. Review these computations. We differentiated the P.S. for $\sin x$ to find P.S for $\cos x$.

Key Fact: If a function f has a Power Series centered at a given a, then the Series must be the Taylor Series (or MacLaurin if $a = 0$).

Here is a chart of the 6 MacLaurin Series that we derived:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$$

Helpful Tip: There are 4 Main Options for finding a functions Power Series representation:

1. Using Substitution into a known Power Series
2. Differentiating a known Power Series.
3. Integrating a known Power Series. Sometimes messier with solving for $+C$.
4. Using the Definition above, sometimes known as the *Chart Method*.
   $\rightarrow$ This option has its limitations if the derivatives are complicated.

Think about which options we have used so far for each function.

Fun Challenges:

- See if you can use 3 options to find the MacLaurin Series for $\cos x$ or $\sin x$.
- See if you can use **all** 4 options to find the MacLaurin Series for $\cosh x$ or $\sinh x$.

Applications We investigated several Applications for Power Series.

- New Series
- New Indefinite Integrals
- New Sums
- New Estimates using the Alternating Series Estimation Theorem
- New Limits

New Series, using substitution into a known Series

Example:

$$x^5 \arctan(3x) = x^5 \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{2n+1} = x^5 \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+6}}{2n+1}}$$

Here need $|3x| < 1$ or $|x| < \frac{1}{3}$, so $\boxed{R = \frac{1}{3}}$.

Example: $$\frac{x}{1+7x} = x\left(\frac{1}{1-(-7x)}\right) = x\sum_{n=0}^{\infty}(-7x)^n = x\sum_{n=0}^{\infty}(-1)^n 7^n x^n = \boxed{\sum_{n=0}^{\infty}(-1)^n 7^n x^{n+1}}$$

Here need $|-7x| < 1$ or $|x| < \frac{1}{7}$, so $\boxed{R = \frac{1}{7}}$.

New Indefinite Integrals

Example: Use Series to compute $\displaystyle\int x^4 e^{-x^2}\ dx$

$$\int \cancelto{\text{P.S.}}{x^4 e^{-x^2}}\, dx = \int x^4 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\ dx = \int x^4 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\ dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{n!}\ dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+5}}{(n!)(2n+5)} + C$$

Note: Here we can leave the $+C$ since we don't have a given function on the left side of the equality chain to plug into to solve for +C.

New Sums: Pattern matching in reverse

Example: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(36)^n\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(6)^{2n}\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!} \cdot \left(\frac{\frac{\pi}{6}}{\frac{\pi}{6}}\right) = \frac{6}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{6}\right)^{2n+1}}{(2n+1)!} = \frac{6}{\pi} \sin\left(\frac{\pi}{6}\right) = \frac{6}{\pi}\left(\frac{1}{2}\right) = \boxed{\frac{3}{\pi}}$$

Example: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 8)^n}{3^{n+1}\ n!} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 8)^n}{3^n\ n!} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 8}{3}\right)^n}{n!}$

$$= \frac{1}{3} e^{\left(\frac{-\ln 8}{3}\right)} = \frac{1}{3} e^{\ln\left(8^{-\frac{1}{3}}\right)} = \frac{1}{3}\left(8^{-\frac{1}{3}}\right) = \frac{1}{3}\left(\frac{1}{8^{\frac{1}{3}}}\right) = \frac{1}{3}\left(\frac{1}{2}\right) = \boxed{\frac{1}{6}}$$

Example: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{2^{4n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(2^2)^{2n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{4^{2n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{4}\right)^{2n}}{(2n)!}$

$$= \cos\left(\frac{\pi}{4}\right) = \boxed{\frac{1}{\sqrt{2}}} = \boxed{\frac{\sqrt{2}}{2}}$$

Example: $\displaystyle\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} - \ldots = \sin\pi = \boxed{0}$

Example: $-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\frac{\pi^8}{8!}-\ldots=(\cos\pi)-1=-1-1=\boxed{-2}$

Example: $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots=-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots\right)=-\ln(1+1))=\boxed{-\ln 2}$

New Estimates for Values of Alternating Series

Example: Estimate $\cos(1)$ with error less than $\frac{1}{100}$. Justify.

$$\cos(1)=\sum_{n=0}^{\infty}\frac{(-1)^n(1)^{2n}}{(2n)!}=1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\ldots=1-\frac{1}{2}+\frac{1}{24}-\frac{1}{720}+\ldots$$

$$\approx 1-\frac{1}{2}+\frac{1}{24}=\frac{24}{24}-\frac{12}{24}+\frac{1}{24}=\boxed{\frac{13}{24}} \quad \leftarrow \text{estimate}$$

Using the Alternating Series Estimation Theorem (ASET), we can approximate the actual sum with only the first three terms as $\frac{13}{24}$, with error *at most* the absolute value of the first neglected term, $\boxed{\frac{1}{6!}}$. Here $\frac{1}{720}<\frac{1}{100}$ as desired.

New Estimates for Definite Integrals

Example: Estimate $\int_0^1 x^3\ln(1+x^3)\,dx$ with error less than $\frac{1}{30}$. Justify.

$$\int_0^1 x^3\ln(1+x^3)\,dx=\int_0^1\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+6}}{n+1}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+7}}{(n+1)(3n+7)}\Bigg|_0^1$$

$$=\frac{x^7}{1\cdot 7}-\frac{x^{10}}{2\cdot 10}+\frac{x^{13}}{3\cdot 13}-\ldots\Bigg|_0^1=\frac{x^7}{7}-\frac{x^{10}}{20}+\frac{x^{13}}{39}-\ldots\Bigg|_0^1$$

$$=\frac{1}{7}-\frac{1}{20}+\frac{1}{39}-\ldots-(0-0+0-\ldots)$$

$$\approx\frac{1}{7}-\frac{1}{20}=\frac{20}{140}-\frac{7}{140}=\boxed{\frac{13}{140}} \quad \leftarrow \quad \textbf{estimate}$$

Using the Alternating Series Estimation Theorem (ASET), we can approximate the actual sum with only the first two terms, and the error from the actual sum will be *at most* the absolute value of the next (first neglected) term, $\frac{1}{39}$. Here $\frac{1}{39}<\frac{1}{30}$ as desired.

Example: **Estimate** $\displaystyle\int_0^1 x\sin\left(x^2\right)\ dx$ with error less than $\dfrac{1}{1000}$. Justify.

$$\int_0^1 x\sin(x^2)\ dx = \int_0^1 x\sum_{n=0}^{\infty}\frac{(-1)^n\left(x^2\right)^{2n+1}}{(2n+1)!}\ dx = \int_0^1 x\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}\ dx$$

$$= \int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(2n+1)!}\ dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+4}}{(2n+1)!(4n+4)}\Bigg|_0^1 = \frac{x^4}{1\cdot 4} - \frac{x^8}{3!\cdot 8} + \frac{x^{12}}{5!\cdot(12)} - \dots\ \Bigg|_0^1$$

$$= \left(\frac{1}{4} - \frac{1}{48} + \frac{1}{1440} - \dots\right) - (0 - 0 + 0 - \dots) \approx \frac{1}{4} - \frac{1}{48} = \boxed{\frac{11}{48}} \quad \leftarrow \textbf{estimate}$$

Using the Alternating Series Estimation Theorem (ASET), we can approximate the actual sum with only the first two terms as $\dfrac{11}{48}$, with error *at most* the absolute value of the first neglected term, $\boxed{\dfrac{1}{1440}}$. Here $\dfrac{1}{1440} < \dfrac{1}{1000}$ as desired.

New Limits using Series

Example: Use series to evaluate the following limit

$$\lim_{x\to 0}\frac{x\cos x}{\sin x} = \lim_{x\to 0}\frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$$

$$= \lim_{x\to 0}\frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)}{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right)} = \lim_{x\to 0}\frac{1 \cancelto{0}{- \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}}{1 \cancelto{0}{- \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}} = \boxed{1}$$

In closing, there are many more subtopics to study for Power Series, but we have a great base of material to build off of for now. Focus on the

- Definitions
- Domains
- Differentiation/Integration
- Derivation of Power Series for Functions, Multiple Methods
- Memorize the 6 Derived MacLaurin Series, study how each one was derived.
- Many Applications