Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121 Final Exam

May 7, 2018

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

- Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or $\cosh(\ln 3)$ should be simplified.

- Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		18
2		22
3		40
4		18
5		26
6		18
7		10
8		10
9		18
10		20
Total		200

1. [18 Points] Evaluate the following **limits**. Please justify your answer. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist. Simplify.

(a) $\lim\limits_{x\to 0} \dfrac{xe^x - \arctan x}{\ln(1+5x) - 5x}$

$$\overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{xe^x + e^x - \dfrac{1}{1+x^2}}{\dfrac{5}{1+5x} - 5} \qquad \overset{0/0}{} \to -(1+x^2)^{-1}$$

$5(1+5x)^{-1} \leftarrow$

$$\underset{L'H}{=} \lim_{x\to 0} \frac{(xe^x + e^x) + e^x + (1+x^2)^{-2}(2x)}{-5(1+5x)^{-2}(5) \quad - 0} = \lim_{x\to 0} \frac{\overset{0}{\cancel{xe^x}} + \boxed{2}\overset{1}{e^x} + \overset{0}{\dfrac{2x}{(1+x^2)^2}}}{\dfrac{(-25)}{\underset{1}{(1+5\underset{0}{\cancel{x}})^2}}} = \boxed{\frac{-2}{25}}$$

(b) Compute $\lim\limits_{x\to 0} \dfrac{xe^x - \arctan x}{\ln(1+5x) - 5x}$ **again** using series.

$$= \lim_{x\to 0} \frac{x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)}{\cancel{(5x)} - \frac{(5x)^2}{2} + \frac{(5x)^3}{3} - \cdots \quad \cancel{-5x}}$$

$$= \lim_{x\to\infty} \frac{\cancel{x+} x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots \cancel{-x} + \frac{x^3}{3} - \frac{x^5}{5} + \ldots}{\frac{-25x^2}{2} + \frac{125x^3}{3} - \ldots} \frac{\left(1/x^2\right)}{\left(\frac{1}{x^2}\right)}$$

or Factor out x^2.

$$= \lim_{x\to 0} \frac{1 + \overset{\to 0}{\frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x}{3} - \frac{x^3}{5} + \ldots}}{\frac{-25}{2} + \underset{0}{\frac{125\cancel{x}}{3} - \ldots}} = \frac{1}{\left(-25/2\right)} = \boxed{\frac{-2}{25}}$$

Match.

1. (Continued) Evaluate the following **limit**. Please justify your answer. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist. Simplify.

$$(c)\quad \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x \overset{1^\infty}{=} \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to\infty}\ln\left[\left(1+\frac{1}{x}\right)^x\right]}$$

$$= e^{\lim_{x\to\infty} \overset{\infty\cdot 0}{x\ln\left(1+\frac{1}{x}\right)}} \ \text{Flip} \ = e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} \quad \frac{0}{0}$$

$$\underset{L'H}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{1+\cancel{\frac{1}{x}}\to 0}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}}\to 1} = e^1 = \boxed{e}$$

2. [22 Points] **Evaluate the following integral.**

(a) $\int \frac{\cos x}{(4+\sin^2 x)^{\frac{5}{2}}}\,dx = \int \frac{1}{\left(\sqrt{4+u^2}\right)^5}\,du = \int \frac{1}{\left(\sqrt{4+4\tan^2\theta}\right)^5}\cdot 2\sec^2\theta\,d\theta$

$\boxed{\begin{array}{l} u=\sin x \\ du=\cos x\,dx \end{array}}$

TRIG. SUB

$\boxed{\begin{array}{l} u=2\tan\theta \\ du=2\sec^2\theta\,d\theta \end{array}}$

$$= \int \frac{1}{\underbrace{\left(\sqrt{4\underbrace{(1+\tan^2\theta)}_{\sec^2\theta}}\right)^5}_{(2\sec\theta)^5}}\cdot 2\sec^2\theta\,d\theta = \int \frac{2\sec^2\theta}{\underbrace{\left(\sqrt{4\sec^2\theta}\right)^5}_{(2\sec\theta)^5}}\,d\theta$$

ISOLATE

$$= \frac{2}{2^5}\int \frac{\sec^2\theta}{\sec^5\theta}\,d\theta = \frac{1}{16}\int \frac{1}{\sec^3\theta}\,d\theta = \frac{1}{16}\int \cos^3\theta\,d\theta = \frac{1}{16}\int \cos^2\theta\,\cos\theta\,d\theta$$

CONVERT

Finish with sub.

$$= \frac{1}{16}\int (1-\sin^2\theta)\cos\theta\,d\theta = \frac{1}{16}\int 1-w^2\,dw = \frac{1}{16}\left[w-\frac{w^3}{3}\right]+C$$

$\boxed{\begin{array}{l} w=\sin\theta \\ dw=\cos\theta\,d\theta \end{array}}$

$$= \frac{1}{16}\left[\sin\theta-\frac{\sin^3\theta}{3}\right]+C$$

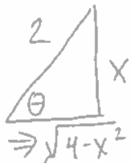
$$= \frac{1}{16}\left[\frac{u}{\sqrt{u^2+4}}-\frac{1}{3}\left(\frac{u}{\sqrt{u^2+4}}\right)^3\right]+C = \boxed{\frac{1}{16}\left[\frac{\sin x}{\sqrt{\sin^2 x+4}}-\frac{1}{3}\,\frac{\sin^3 x}{(\sin^2 x+4)^{3/2}}\right]+C}$$

2. (Continued) Evaluate each of the following **integrals**.

(b) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2\theta}{\sqrt{\underbrace{4-4\sin^2\theta}_{4(1-\sin^2\theta)}}} \cdot 2\cos\theta d\theta = \int \frac{4\sin^2\theta}{\underbrace{\sqrt{4\cos^2\theta}}_{\cancel{2\cos\theta}}} \cdot \cancel{2\cos\theta}\, d\theta$

TRIG SUB

$$\boxed{\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta d\theta \end{array}}$$



$$= 4\int \sin^2\theta\, d\theta = \cancel{4}\int \frac{1-\cos(2\theta)}{\cancel{2}}\, d\theta = 2\int 1-\cos(2\theta) d\theta = 2\left[\theta - \frac{\overbrace{\sin(2\theta)}^{\cancel{2}\sin\theta\cos\theta}}{\cancel{2}}\right] + C$$

$$= \boxed{2\left[\arcsin\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)\frac{\sqrt{4-x^2}}{2}\right] + C}$$

(c) $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \arcsin\left(\frac{x}{2}\right)\Big|_1^{\sqrt{3}} = \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right)$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \boxed{\frac{\pi}{6}}$$

3. [40 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_6^7 \frac{8}{x^2-4x-12}\,dx = \int_6^7 \frac{8}{(X-6)(X+2)}\,dX = \lim_{t\to6^+}\int_t^7 \frac{8}{(X-6)(X+2)}\,dX$

PFD

$$\left[\frac{8}{(X-6)(X+2)} = \frac{A}{X-6} + \frac{B}{X+2}\right](X-6)(X+2)$$

$$8 = A(X+2) + B(X-6)$$

$$= (A+B)X + 2A - 6B$$

- $A+B=0 \Rightarrow B=-A$
- $2A-6B=8 \quad 2A+6A=8$

$8A=8$

$\boxed{A=1 \Rightarrow B=-1}$

$$= \lim_{t\to6^+}\int_t^7 \frac{1}{X-6} - \frac{1}{X+2}\,dX$$

$$= \lim_{t\to6^+} \ln|X-6| - \ln|X+2| \Big|_t^7$$

$$= \lim_{t\to6^+} \overset{\to 0}{\cancel{\ln 1}} - \overset{\text{Finite}}{\ln 9} \boxed{-} \Big(\underset{\boxed{-\infty}}{\overset{0^+}{\cancel{\ln|t-6|}}} - \overset{-\ln 8 \ \text{Finite}}{\cancel{\ln|t+2|}}\Big)$$

$$= -(-\infty) = \boxed{+\infty} \quad \text{Diverges}$$

(b) $\displaystyle\int_7^\infty \frac{8}{x^2-4x-12}\,dx$ **Tip:** Reuse your algebra work from part (a)

See Above PFD

$$= \lim_{t\to\infty}\int_7^t \frac{8}{(X-6)(X+2)}\,dX = \lim_{t\to\infty}\int_7^t \frac{1}{X-6} - \frac{1}{X+2}\,dX = \lim_{t\to\infty} \ln|X-6| - \ln|X+2| \Big|_7^t$$

Indeterminate

$$= \lim_{t\to\infty} \overset{\infty}{\cancel{\ln|t-6|}} - \overset{\infty}{\cancel{\ln|t+2|}} - \Big(\overset{0}{\cancel{\ln 1}} - \ln 9\Big)$$

$$= \lim_{t\to\infty} \ln\left|\frac{t-6}{t+2}\right| + \ln 9 = \lim_{t\to\infty} \overset{0}{\cancel{\ln}} \left|\overset{1}{\cancel{\frac{1-6/t}{1+2/t}}}\right| + \ln 9 = \boxed{\ln 9} \quad \text{Converges}$$

3. (Continued) For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(c) $\displaystyle\int_0^{e^3} \frac{1}{x\,[3+(\ln x)^2]}\,dx = \lim_{t\to 0^+} \int_t^{e^3} \frac{1}{x\,[3+(\ln x)^2]}\,dx$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} x=t &\Rightarrow u=\ln t \\ x=e^3 &\Rightarrow u=\ln e^3 = 3 \end{aligned}}$$

$$= \lim_{t\to 0^+} \int_{\ln t}^{3} \frac{1}{3+u^2}\,du$$

$$= \lim_{t\to 0^+} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{\ln t}^{3}$$

$$= \lim_{t\to 0^+} \frac{1}{\sqrt{3}}\left[\underbrace{\arctan\left(\overset{\sqrt{3}}{\frac{3}{\sqrt{3}}}\right)}_{\pi/3} - \underbrace{\arctan\left(\frac{\overset{-\infty}{\ln t}}{\sqrt{3}}\right)}_{(-\pi/2)}\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{3}+\frac{\pi}{2}\right] = \frac{1}{\sqrt{3}}\left[\frac{2\pi}{6}+\frac{3\pi}{6}\right] = \boxed{\frac{5\pi}{6\sqrt{3}}}$$

$\frac{x^{3/2}}{x} = x^{1/2}$

3. (Continued) For the following **improper integral**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(d) $\int_0^1 \sqrt{x}\,\ln x\,dx = \lim_{t\to 0^+} \int_t^1 \sqrt{x}\,\ln x\,dx = \lim_{t\to 0^+} \frac{2}{3}x^{3/2}\ln x\Big|_t^1 - \frac{2}{3}\int_t^1 x^{1/2}\,dx$

$$\boxed{\begin{array}{ll} u=\ln x & dv=\sqrt{x}\,dx \\ du=\frac{1}{x}dx & v=\frac{2}{3}x^{3/2} \end{array}}$$

$$= \lim_{t\to 0^+} \frac{2}{3}x^{3/2}\ln x\Big|_t^1 - \frac{4}{9}x^{3/2}\Big|_t^1$$

$$= \lim_{t\to 0^+} \overset{\to 0}{\cancel{\frac{2}{3}\ln 1}} - \overset{0\cdot(-\infty)}{\cancel{\frac{2}{3}t^{3/2}\ln t}} - \frac{4}{9}\left(1 - \overset{\to 0}{\cancel{t^{3/2}}}\right)$$

Indeterminate

(*) See Below $\to 0$

$$= \boxed{\frac{-4}{9}}$$ Converges.

$$(*)\ \lim_{t\to 0^+} \overset{0\,(-\infty)}{t^{3/2}\cdot\ln t} = \lim_{t\to 0^+} \overset{-\infty/\infty}{\frac{\ln t}{\left(1/t^{3/2}\right)}} \underset{L'H}{=} \lim_{t\to 0^+} \frac{1/t}{\frac{-3}{2t^{5/2}}} \overset{\frac{-2t^{5/2}}{3}}{\nearrow} = \lim_{t\to 0^+} \frac{-2\overset{\to 0}{\cancel{t^{3/2}}}}{3} = 0$$

4. [18 Points] Find the sum of each of the following series (which do converge). Simplify.

(a) Geometric

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, 5^{2n+1}}{2^{5n-1}} = \frac{-5^3}{2^4} + \frac{5^5}{2^9} - \frac{5^7}{2^{14}} + \dots$$

$$a = \frac{-5^3}{2^4} = \frac{-125}{16}$$

$$r = \frac{-5^2}{2^5} = \frac{-25}{32}$$

$$\text{Sum} = \frac{a}{1-r} = \frac{\frac{-125}{16}}{1-\left(-\frac{25}{32}\right)} = \frac{\frac{-125}{\cancel{16}}}{\frac{57}{32}} \quad \frac{\overset{2}{\cancel{32}}}{57} = \boxed{\frac{-250}{57}}$$

$$\frac{32}{32}$$

(b)

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,(\ln 9)^n}{2^{n+1}\cdot n!} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(\ln 9)^n}{2^n\, n!} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{-\ln 9}{2}\right)^n}{n!}$$

$$= -\frac{1}{2}e^{\frac{-\ln 9}{2}} = -\frac{1}{2}\cancel{e}^{\cancel{\ln}\left[9^{-1/2}\right]} = -\frac{1}{2}\cdot\frac{1}{\sqrt{9}} = -\frac{1}{2}\cdot\frac{1}{3} = \boxed{\frac{-1}{6}}$$

(c)

$$\sum_{n=0}^{\infty} \frac{(-1)^n\pi^{2n+1}}{9^n\,(2n)!} = \pi\sum_{n=0}^{\infty}\frac{(-1)^n\,\pi^{2n}}{3^{2n}(2n)!} = \pi\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{3}\right)^{2n}}{(2n)!}$$

$$= \pi\, \overset{1/2}{\cancel{\cos\left(\frac{\pi}{3}\right)}} = \boxed{\frac{\pi}{2}}$$

(d)

$$-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots = \overset{\pi/4}{\cancel{(\arctan 1)}} - 1 = \boxed{\frac{\pi}{4}-1}$$

(e)

$$-\frac{\pi^3}{3!}+\frac{\pi^5}{5!}-\frac{\pi^7}{7!}+\frac{\pi^9}{9!}-\dots = \overset{0}{\cancel{(\sin\pi)}} - \pi = \boxed{-\pi}$$

(f)

$$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\dots = -\left(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots\right) = -\ln(1+1) = \boxed{-\ln 2}$$

5. [26 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(7n)}{n^7+7}$ $\overset{A.S.}{\hookrightarrow}$ $\sum_{n=1}^{\infty} \frac{\arctan(7n)}{n^7+7}$

Bound Terms: $\frac{\arctan(7n)}{n^7+7} \leq \frac{\pi/2}{n^7+7} \leq \frac{\pi/2}{n^7}$ and $\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^7}$ Converges because Constant Multiple of Convergent p-Series $p=7>1$ is Convergen

$\Rightarrow$ A.S. Converges by CT

$\Rightarrow$ O.S. $\boxed{A.C.}$ (by Definition)

(b) $\sum_{n=1}^{\infty} \arctan\left(\frac{n^7+1}{n^7+7}\right)$ Diverges by nTDT because

$$\lim_{n\to\infty} \arctan\left(\frac{n^7+1}{n^7+7}\right) = \lim_{n\to\infty} \arctan\left(\frac{1+\cancel{1/n^7}^{\to 0}}{1+\cancel{7/n^7}^{\to 0}}\right) = \arctan 1 = \frac{\pi}{4} \neq 0.$$

5. (Continued) Determine whether the given series is **absolutely convergent, conditionally convergent**, or **divergent**. Justify your answers.

(c) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n+1}{n^2}\right)$

A.S. $\sum_{n=1}^{\infty}\frac{n+1}{n^2} \approx \sum_{n=1}^{\infty}\frac{n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n}$ Divergent (Harmonic) p-Series $p=1$

Bound Terms:

$\frac{n+1}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}$ and

$\Rightarrow$ A.S. Diverges by CT (Could also use LCT)

AST on O.S.

① $b_n = \frac{n+1}{n^2} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n+1}{n^2}\frac{(1/n^2)}{(1/n^2)} = \lim_{n\to\infty}\frac{\cancelto{0}{1/n + 1/n^2}}{1} = 0$

③ Terms decreasing

$f(x) = \frac{x+1}{x^2}$ has $f'(x) = \frac{x^2(1)-(x+1)(2x)}{x^4}$

$= \frac{-x^2-2x}{x^4} < 0$ for $x>0$ ✓

O.S. Converges by AST

O.S. $\boxed{C.C.}$

(by Definition).

5. (Continued) Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3n)!\,\ln n}{(n!)^2\,e^{4n}\,n^n}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,[3(n+1)]!\,\ln(n+1)}{[(n+1)!]^2\,e^{4(n+1)}\,(n+1)^{n+1}}}{\dfrac{\cancel{(-1)^{n}}\,(3n)!\,\ln n}{(n!)^2\,e^{4n}\,n^n}}\right|$$

$$= \lim_{n\to\infty} \frac{\overset{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}{(3n+3)!}}{\cancel{(3n)!}} \cdot \underbrace{\cancel{\frac{\ln(n+1)}{\ln n}}}_{\substack{(*)\\ \text{See Below}}}{}^{\to 1} \cdot \frac{\cancel{(n!)^2}}{\underset{(n+1)^2\cancel{(n!)^2}}{[(n+1)!]^2}} \cdot \frac{e^{4n}}{e^{4n+4}} \cdot \boxed{\frac{n^n}{\underset{(n+1)^n(n+1)}{(n+1)^{n+1}}}} \to 1/e$$

why?

$$= \lim_{n\to\infty} \left(\frac{3n+3}{n+1}\right)^{\to 3}\left(\frac{3n+2}{n+1}\right)^{\to 3}\left(\frac{3n+1}{n+1}\right)^{\to 3} \cdot \frac{1}{e^4} \cdot \frac{1}{e}$$

$$= \frac{27}{e^5} < 1 \quad \text{o.s. } \boxed{\text{A.C.}} \text{ by Ratio Test.}$$

$$(*)\ \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} \underset{\text{L'H}}{\overset{\infty/\infty}{=}} \lim_{x\to\infty} \frac{1}{1} = 1$$

5. (Continued) Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(e) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+5}{n^5+2}$ $\xrightarrow{A.S.}$ $\sum_{n=1}^{\infty} \frac{n^2+5}{n^5+2} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$

Converges p-Series

$p = 3 > 1$

LCT

$$\lim_{n\to\infty} \frac{\frac{n^2+5}{n^5+2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^5+5n^3}{n^5+2} \frac{\left(\frac{1}{n^5}\right)}{\left(\frac{1}{n^5}\right)} = \lim_{n\to\infty} \frac{1+\cancel{\frac{5}{n^2}}^{\to 0}}{1+\cancel{\frac{2}{n^5}}^{\to 0}} = 1$$

Finite, Non-Zero

$\Rightarrow$ A.S. Converges by LCT

$\Rightarrow$ O.S. $\boxed{\text{A.C.}}$ (by Definition)

6. [18 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x-4)^n}{n^2\cdot 5^n}$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(3x-4)^{n+1}}{(n+1)^2\,5^{n+1}}}{\frac{\cancel{(-1)^n}\,(3x-4)^n}{n^2\cdot 5^n}}\right|=\lim_{n\to\infty}\left|\frac{(3x-4)^{n+1}}{(3x-4)^n}\right|\cdot\frac{n^2}{(n+1)^2}\cdot\frac{5^n}{5^{n+1}}$$

$$=\lim_{n\to\infty}|3x-4|\overset{\to 1}{\cancel{\left(\frac{n}{n+1}\right)^2}}\cdot\frac{1}{5}$$

$$=\frac{|3x-4|}{5}<1$$ ← Converges by Ratio Test when

$|3x-4|<5$

$-5<3x-4<5$

$+4 \quad +4 \quad +4$

$-1<3x<9$

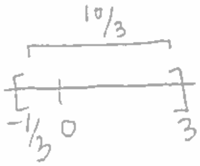
$-\frac{1}{3}<x<3$

Endpoints: $x=3$ O.S. becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\overbrace{(9-4)^n}^{5^n}}{n^2\cdot \cancel{5^n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\xrightarrow{A.S.}\sum_{n=1}^{\infty}\frac{1}{n^2}$ Converges p-Series $p=2>1$

$\Rightarrow$ O.S. Converges by ACT (can also use AST)

$x=-\frac{1}{3}$ O.S. becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\overbrace{\left[3(-\tfrac{1}{3})-4\right]^n}^{\overbrace{(-5)^n}^{(-1)^n5^n}}}{n^2\cdot 5^n}$

$\displaystyle=\sum_{n=1}^{\infty}\frac{\overset{1}{\cancel{(-1)^{2n}}}\,\cancel{5^n}}{n^2\,\cancel{5^n}}=\sum_{n=1}^{\infty}\frac{1}{n^2}$ Converges p-Series $p=2>1$



$I=\boxed{\left[-\frac{1}{3},3\right]}$

$R=\boxed{\frac{5}{3}}$

6. (Continued) Find the **Interval** and **Radius** of Convergence for each of the following power series. Analyze carefully and with full justification.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!}$

<u>Ratio Test</u>

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}\, x^{2(n+1)}}{[2(n+1)]!}}{\frac{\cancel{(-1)^n}\, x^{2n}}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{x^{2n}}\right| \cdot \frac{\cancel{(2n)!}}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)\cancel{(2n)!}}} = \lim \frac{|x|^2}{(2n+2)(2n+1)} = 0 < 1$$

Regardless of x

Converges by R.T. for all x

$I = \boxed{(-\infty, \infty)} = \mathbb{R}$

$R = \boxed{\infty}$

(c) $\sum_{n=1}^{\infty} n!\,(x-6)^n$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\overbrace{(n+1)!}^{(n+1)\cancel{n!}}\,(x-6)^{n+1}}{\cancel{n!}\,(x-6)^n}\right| = \lim_{n\to\infty} (n+1)|x-6| = \infty > 1$$

Diverges by Ratio Test <u>unless</u> $x=6$.

$I = \boxed{\{6\}}$

$R = \boxed{0}$

7. [10 Points] **Please analyze with detail and justify carefully. Simplify.**

(a) Use MacLaurin series to **Estimate** $\int_0^1 x\sin\left(x^2\right)\, dx$ with error less than $\frac{1}{1000}$.

$$=\int_0^1 x \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}\, dx = \int_0^1 x \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\, dx$$

$$\begin{array}{r} 120 \\ \underline{12} \\ 240 \\ \underline{1200} \\ 1440 \end{array}$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)!\,(4n+4)}\Bigg|_0^1$$

$$= \frac{x^4}{4} - \frac{x^8}{3!\cdot 8} + \frac{x^{12}}{5!\cdot(12)} - \cdots \Bigg|_0^1 = \frac{1}{4} - \frac{1}{48} + \frac{1}{1440} - \cdots - \cancelto{0}{\left(0-0+0\pm\cdots\right)}$$

$$\approx \frac{1}{4} - \frac{1}{48} = \frac{12}{48} - \frac{1}{48} = \boxed{\frac{11}{48}} \leftarrow \text{Estimate}$$

Using ASET, we can estimate the Full Sum using only the first two terms with error at most |the first neglected term|

$\frac{1}{1440} < \frac{1}{1000}$ as desired.

(b) Use MacLaurin Series to **Estimate** $\frac{1}{\sqrt{e}}$ with error less than $\frac{1}{100}$.

$$\begin{array}{r} 16 \\ \underline{16} \\ 96 \\ \underline{160} \\ 256 \end{array}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

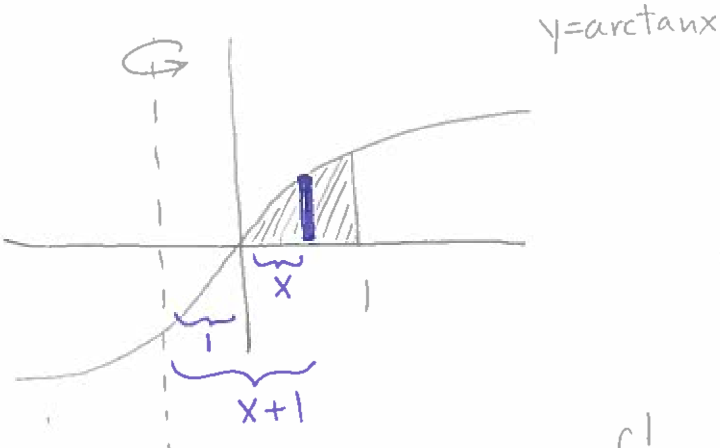
$$\frac{1}{\sqrt{e}} = e^{-1/2} = 1 + (-1/2) + \frac{(-1/2)^2}{2!} + \frac{(-1/2)^3}{3!} + \frac{(-1/2)^4}{4!} + \cdots$$

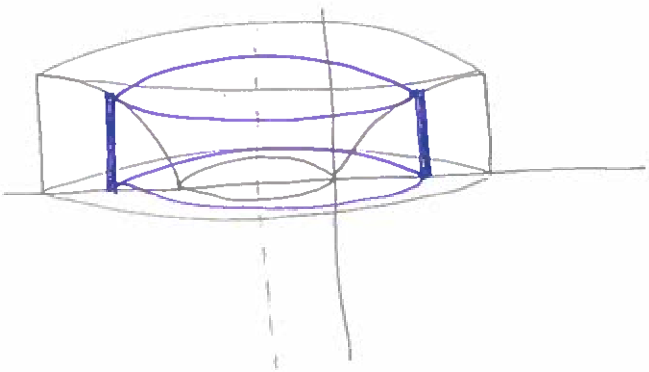
$$= 1 - 1/2 + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} - \cdots$$

$$\approx 1 - \frac{1}{2} + \frac{1}{8} - 1/48 = \frac{48}{48} - \frac{24}{48} + \frac{6}{48} - \frac{1}{48} = \boxed{\frac{29}{48}} \leftarrow \text{Estimate}$$

Using ASET, we can estimate the full sum using only the first 4 terms with error at most the |first neglected term| $\frac{1}{384} < \frac{1}{100}$ as desired.

8. [10 Points] Consider the region bounded by $y = \arctan x$, $y = 0$, $x = 0$ and $x = 1$. Rotate the region about the vertical line $\boxed{x = -1}$. **COMPUTE** the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.





$$\text{Volume} = 2\pi\int_0^1 \text{Radius}\cdot\text{Height}\,dx$$

$$= 2\pi\int_0^1 (x+1)\arctan x\,dx$$

IBP

$u=\arctan x \quad dv = x+1\,dx$

$du=\frac{1}{1+x^2}dx \quad v=\frac{x^2}{2}+x$

$$=2\pi\left[\left(\frac{x^2}{2}+x\right)\arctan x\Big|_0^1 - \int_0^1\left(\frac{x^2}{2}+x\right)\left[\frac{1}{1+x^2}\right]dx\right]$$

slip-in|slip-out

$$=2\pi\left[\left(\frac{x^2}{2}+x\right)\arctan x\Big|_0^1 - \frac12\int_0^1\frac{x^2{}^{+1-1}}{1+x^2}dx - \int_0^1\frac{x}{1+x^2}dx\right.$$

$$\downarrow \qquad -\frac12\int_0^1 1-\frac{1}{1+x^2}dx$$

$$=2\pi\left[\left(\frac{x^2}{2}+x\right)\arctan x - \frac12\left(x-\arctan x\right)\Big|_0^1 - \frac12\ln|1+x^2|\Big|_0^1\right.$$

$$=2\pi\left[\frac32\,\overset{\pi/4}{\cancel{\arctan}\,1} - \frac12\left(1-\overset{\pi/4}{\cancel{\arctan}\,1}\right) - (0-0) - \frac12\ln 2 + \overset{0}{\cancel{\ln 1}}\right]$$

$$=\boxed{2\pi\left[\underbrace{2\left(\frac{\pi}{4}\right)}_{\pi/2}-\frac12-\frac{\ln 2}{2}\right]} \qquad \text{OR} \quad \pi^2-\pi-\pi\ln 2 = \pi(\pi-1-\ln 2)$$

9. [18 Points]

(a) Consider the Parametric Curve represented by $x = \ln t + \ln(1-t^2)$ and $y = \sqrt{8}\arcsin t$.

COMPUTE the **arclength** of this parametric curve for $\frac{1}{4} \le t \le \frac{1}{2}$. Show that the answer simplifies to $\boxed{\ln\left(\frac{5}{2}\right)}$

$$\frac{dx}{dt} = \frac{1}{t} - \frac{2t}{1-t^2} \qquad \frac{dy}{dt} = \frac{\sqrt{8}}{\sqrt{1-t^2}}$$

$$L = \int_{1/4}^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_{1/4}^{1/2} \sqrt{\left(\frac{1}{t} - \frac{2t}{1-t^2}\right)^2 + \left(\frac{\sqrt{8}}{\sqrt{1-t^2}}\right)^2}\,dt = \int_{1/4}^{1/2} \sqrt{\frac{1}{t^2} \underbrace{- \frac{4}{1-t^2} + \frac{4t^2}{(1-t^2)^2} + \frac{8}{1-t^2}}}\,dt$$

$$= \int_{1/4}^{1/2} \sqrt{\frac{1}{t^2} + \frac{4}{1-t^2} + \frac{4t^2}{(1-t^2)^2}}\,dt = \int_{1/4}^{1/2} \sqrt{\left(\frac{1}{t} + \frac{2t}{1-t^2}\right)^2}\,dt$$

$$= \int_{1/4}^{1/2} \frac{1}{t} + \frac{2t}{1-t^2}\,dt = \ln|t| - \ln(1-t^2)\Big|_{1/4}^{1/2}$$

$$= \ln\left(\tfrac{1}{2}\right) - \ln\left(\tfrac{3}{4}\right) - \left(\ln\left(\tfrac{1}{4}\right) - \ln\left(\tfrac{15}{16}\right)\right)$$

Simplify.

$$= \overset{0}{\cancel{\ln 1}} - \ln 2 - \ln 3 + \ln 4 - \left(\overset{0}{\cancel{\ln 1}} - \ln 4 - \ln 15 + \ln 16\right)$$

$$= -\ln 2 - \ln 3 + \underbrace{2\ln 4}_{\cancel{\ln 16}} + \ln 15 - \cancel{\ln 16}$$

$$= -\ln 2 \underbrace{- \ln 3 + \ln 15}$$

$$= -\ln 2 + \ln\left(\tfrac{15}{3}\right)$$

$$= -\ln 2 + \ln 5 = \boxed{\ln\left(\tfrac{5}{2}\right)}$$ ✓

OR $\ln\left(\frac{1/2}{3/4}\right)^{4/3} - \ln\left(\frac{1/4}{15/16}\right)^{16/15}$

$= \ln\left(\frac{2}{3}\right) - \ln\left(\frac{4}{15}\right)$

$= \ln\left(\frac{2/3}{4/15}\right)^{15/4}$

$= \boxed{\ln\left(\frac{5}{2}\right)}$ ✓

OR $= \underbrace{\ln\left(\tfrac{1}{2}\right) - \ln\left(\tfrac{3}{4}\right) - \ln\left(\tfrac{1}{4}\right) + \ln\left(\tfrac{15}{16}\right)}$

$= \ln\left(\frac{15}{32}\right) - \ln\left(\frac{3}{16}\right)$

$= \ln\left(\frac{15/32}{3/16}\right)^{16/3} = \boxed{\ln\left(\frac{5}{2}\right)}$ ✓

9. (Continued)

(b) Consider a different Parametric Curve represented by $x = t - e^{2t}$ and $y = 1 - \sqrt{8}e^t$.

COMPUTE the **surface area** obtained by rotating this curve about the y-axis for $0 \leq t \leq 1$.

Simplify. Show that the answer simplifies to $\boxed{2\pi\left(2 - \frac{e^4}{2}\right)}$

$$\frac{dx}{dt} = 1 - 2e^{2t} \quad \frac{dy}{dt} = -\sqrt{8}e^t$$

$$S.A. = 2\pi\int_0^1 x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2\pi\int_0^1 (t - e^{2t})\sqrt{(1-2e^{2t})^2 + (-\sqrt{8}e^t)^2}\,dt$$

$$= 2\pi\int_0^1 (t - e^{2t})\sqrt{1 - \underbrace{4e^{2t} + 4e^{4t} + 8e^{2t}}_{+4e^{2t}}}\,dt = 2\pi\int_0^1 (t - e^{2t})\sqrt{1 + 4e^{2t} + 4e^{4t}}\,dt$$

$$= 2\pi\int_0^1 (t - e^{2t})\cancel{\sqrt{(1+2e^{2t})^2}}\,dt = 2\pi\int_0^1 (t - e^{2t})(1 + 2e^{2t})\,dt$$

$$= 2\pi\int_0^1 t - e^{2t} + 2\underbrace{te^{2t}}_{IBP(*)} - 2e^{4t}\,dt = 2\pi\left[\frac{t^2}{2} - \frac{e^{2t}}{2} + 2\left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4}\right) - \frac{\cancel{2}e^{4t}}{\cancel{4}_{2}}\right]\Bigg|_0^1$$

$$= 2\pi\left[\frac{t^2}{2} - \frac{e^{2t}}{2} + te^{2t} - \frac{e^{2t}}{2} - \frac{e^{4t}}{2}\right]\Bigg|_0^1$$

$$= 2\pi\left[\frac{1}{2} \overset{\text{cancel}}{\cancel{- \frac{e^2}{2} + e^2 - \frac{e^2}{2}}} - \frac{e^4}{2} - \left(0 - \overset{1/2}{\cancel{\frac{e^0}{2}}} + 0 - \overset{1/2}{\cancel{\frac{e^0}{2}}} - \overset{1/2}{\cancel{\frac{e^0}{2}}}\right)\right] = 2\pi\left[\frac{1}{2} - \frac{e^4}{2} + \frac{3}{2}\right]$$

$$= \boxed{2\pi\left[2 - e^4/2\right]}$$

✓

$$\boxed{\begin{array}{ll} u = t & dv = e^{2t}dt \\ du = dt & v = \frac{e^{2t}}{2} \end{array}}$$

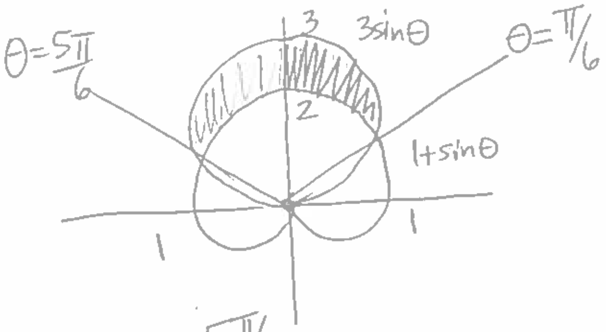
$$(*)\int te^{2t}dt = \frac{te^{2t}}{2} - \frac{1}{2}\int e^{2t}dt$$

$$= \frac{te^{2t}}{2} - \frac{e^{2t}}{4} + C$$

10. [20 Points] For each of the following problems, do the following two things:

1. Sketch the Polar curves and shade the described bounded region.
2. Set-Up but **DO NOT EVALUATE** the Integral representing the area of the described bounded region.

(a) The area bounded outside the polar curve $r = 1+\sin\theta$ and inside the polar curve $r = 3\sin\theta$.



Intersect? $1+\sin\theta = 3\sin\theta$

$1 = 2\sin\theta$

$\sin\theta = 1/2$

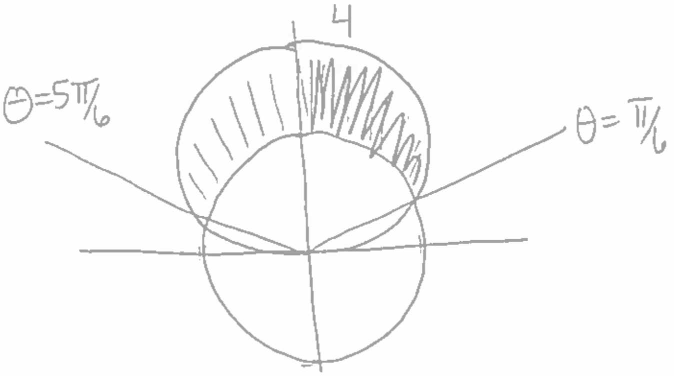
$\theta = \pi/6, 5\pi/6$

$$\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 d\theta$$

OR $$= \boxed{2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (3\sin\theta)^2 - (1+\sin\theta)^2 d\theta\right]}$$

↑
Double by Symmetry

(b) The area bounded outside the polar curve $r = 2$ and inside the polar curve $r = 4\sin\theta$.



Intersect?

$4\sin\theta = 2$

$\sin\theta = 1/2$

$\theta = \pi/6, 5\pi/6$

$$\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 d\theta$$

OR $$= \boxed{2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (4\sin\theta)^2 - (2)^2 d\theta\right]}$$

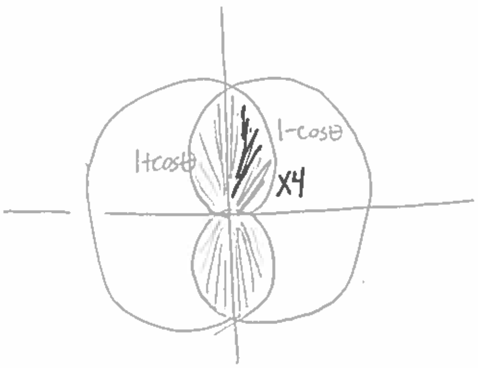
↑
Double by Symmetry

10. (Continued) For the following problem, do the following two things:

1. Sketch the Polar curves and shade the described bounded region.

2. Set-Up but **DO NOT EVALUATE** the Integral representing the area of the described bounded region.

(c) The **area** that lies inside both of the curves $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

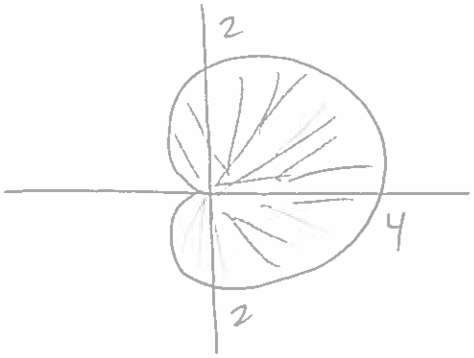


$$\text{Area} = 4\left(\tfrac{1}{2}\right)\int_0^{\pi/2} (\text{Radius})^2\, d\theta$$

$$\boxed{4\left(\tfrac{1}{2}\right)\int_0^{\pi/2} (1 - \cos\theta)^2\, d\theta}$$

By Symmetry

(d) The **area** bounded inside the polar curve $r = 2 + 2\cos\theta$.



$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (\text{Radius})^2\, d\theta$$

$$= \boxed{\frac{1}{2}\int_0^{2\pi} (2 + 2\cos\theta)^2\, d\theta}$$