Extra Examples of Interval and Radius of Convergence

Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x-4)^n}{5^n\sqrt{n}}$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}(3x-4)^{n+1}}{5^{n+1}\sqrt{n+1}}}{\dfrac{\cancel{(-1)^n}(3x-4)^n}{5^n\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\frac{(3x-4)^{n+1}}{(3x-4)^n}\right|\cdot\frac{5^n}{\underbrace{5^{n+1}}_{\text{extra }5}}\cdot\frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|3x-4|}{5} < 1$$

$|3x-4|$ ← Don't Drop

$\sqrt{\frac{n}{n+1}} \to 1$

Converges by R.T. when $L<1$. — Must State this Language.

$$\frac{|3x-4|}{5} < 1 \Rightarrow |3x-4| < 5 \Rightarrow -5 < 3x-4 < 5$$

$+4 \quad +4 \quad +4$

$$\frac{-1 < 3x < 9}{3} \qquad -\frac{1}{3} < x < 3$$

Manually Test Convergence at Endpoints ← Doing This because we need to figure out the 2 endpoint Cases (INCONCLUSIVE) where R.T. has $L=1$

Take $x=3$. Original Series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n[3(3)-4]^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n\cancel{5^n}}{\cancel{5^n}\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

Converges by AST ✓

① $b_n = \frac{1}{\sqrt{n}}$

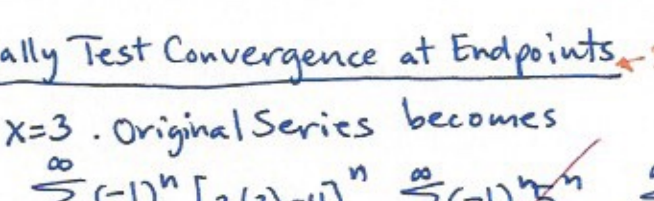
② $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

③ $b_{n+1} = \frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}} = b_n$ Terms Decreasing

Take $x=-\frac{1}{3}$. Original Series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n(-1-4)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n\overbrace{(-5)^n}^{(-1)^n5^n}}{\cancel{5^n}\sqrt{n}} = \sum_{n=1}^{\infty}\frac{\overbrace{(-1)^n(-1)^n}^{(-1)^{2n}\text{ Even} = 1}}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

Diverges p-Series $p=\frac{1}{2}<1$



$$\boxed{\begin{aligned} I &= (-\tfrac{1}{3}, 3] \\ R &= \tfrac{5}{3} \end{aligned}}$$

Find the **Interval** and **Radius** of Convergence for each of the following power series. Analyze carefully and with full justification.

center point

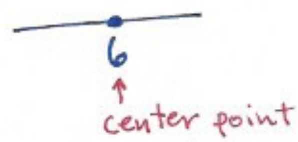
(b) $\displaystyle\sum_{n=1}^{\infty} n!\,(x-6)^n$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\overbrace{(n+1)!}^{(n+1)n!}(x-6)^{n+1}}{\cancel{n!}\,(x-6)^n}\right|=\lim_{n\to\infty}\overbrace{(n+1)}^{\infty}|x-6|=\infty>1$$

Diverges by R.T. forall x, UNLESS x=6

$$\boxed{\begin{array}{c} I=\{6\} \\ R=0 \end{array}}$$



(c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}}{[2(n+1)]!}}{\dfrac{x^n}{(2n)!}}\right|=\lim_{n\to\infty}\overbrace{\left|\frac{x^{n+1}}{x^n}\right|}^{|x|}\cdot\frac{\cancel{(2n)!}}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)(2n)!}}=\lim_{n\to\infty}\frac{|x|}{\underbrace{(2n+2)(2n+1)}_{\infty}}=0<1$$

Converges by R.T. forall Real x.

$$\boxed{\begin{array}{c} I=(-\infty,\infty) \\ R=\infty \end{array}}$$