Quiz #1 Final Answers.

1a. Prove $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

Let $y = \arctan x$

Invert $\boxed{\tan y = x}$

Differentiate both sides $\frac{d}{dx}\left[\tan y\right] = \frac{d}{dx}\left[x\right]$

$$\sec^2 y \cdot \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{\sec^2 y}$

Identity to Finish

$$= \frac{1}{1+\tan^2 y}$$

$$= \frac{1}{1+(\tan y)^2} = \frac{1}{1+x^2} \checkmark$$

x

1b. $$\int \frac{1}{25+x^2}\,dx = \int \frac{1}{25\left(1+\frac{x^2}{25}\right)}\,dx$$

$$= \frac{1}{25}\int \frac{1}{1+\frac{x^2}{25}}\,dx$$

Note: Can also do L.I.D.S. proof in other direction, but you should know this algebra option in case you're forced to use Integration

$$u = \frac{x}{5}$$
$$du = \frac{1}{5}dx$$
$$5du = dx$$

$$= \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx$$

$$= \frac{5}{25}\int \frac{1}{1+u^2}\,du \qquad \text{yes!}$$

$$= \frac{1}{5}\arctan u + C$$

$$= \boxed{\frac{1}{5}\arctan\left(\frac{x}{5}\right) + C}$$

2. $$\int_2^{2\sqrt{3}} \frac{1}{\sqrt{16-x^2}} + \frac{1}{4+x^2}\,dx = \arcsin\left(\frac{x}{4}\right) + \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Bigg|_2^{2\sqrt{3}}$$

no $1/a$ here.

a-rules.

$$= \arcsin\left(\frac{2\sqrt{3}}{4}\right) + \frac{1}{2}\arctan\left(\frac{2\sqrt{3}}{2}\right) - \left[\arcsin\left(\frac{2}{4}\right) + \frac{1}{2}\arctan\left(\frac{2}{2}\right)\right]$$

$\frac{\sqrt{3}}{2}$ $\sqrt{3}$ $\frac{1}{2}$ 1

$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\arctan(\sqrt{3}) - \arcsin\left(\frac{1}{2}\right) - \frac{1}{2}\arctan(1)$$

$\pi/3$ $\pi/3$ $\pi/6$ $\pi/4$

$$= \pi/3 + \cancel{\frac{\pi}{6}} - \cancel{\pi/6} - \pi/8$$

cancel

$$= \pi/3 - \pi/8 = \frac{8\pi}{24} - \frac{3\pi}{24} = \boxed{\frac{5\pi}{24}}$$

3. $$\int_{-\ln 2}^{-\ln\left(\frac{2}{\sqrt{3}}\right)} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int_{1/2}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-u^2}}\,du$$

$e^{2x} \to (e^x)^2$

create perfect square

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}}$$

$$x = -\ln 2 \Rightarrow u = e^{-\ln 2} = \cancel{e}^{\cancel{\ln}(2^{-1})} = \frac{1}{2}$$

$$x = -\ln\left(\frac{2}{\sqrt{3}}\right) \Rightarrow u = e^{-\ln\left(\frac{2}{\sqrt{3}}\right)} = \cancel{e}^{\cancel{\ln}\left[\left(\frac{2}{\sqrt{3}}\right)^{-1}\right]} = \frac{\sqrt{3}}{2}$$

$$= \arcsin u \Big|_{1/2}^{\sqrt{3}/2}$$

$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right)$$

$\pi/3$ $\qquad$ $\pi/6$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \boxed{\frac{\pi}{6}}$$ Match!

$\frac{2\pi}{6}$

4. $$\int \frac{x^2}{x^2+3}\,dx = \int \frac{x^2+3-3}{x^2+3}\,dx = \int \frac{\cancel{x^2+3}}{\cancel{x^2+3}} - \frac{3}{x^2+3}\,dx$$

Slip-in/Slip-out $\qquad$ 1

$$= x - 3\int \frac{1}{x^2+3}\,dx$$ "a-rule"

$$= x - 3\left(\frac{1}{\sqrt{3}}\right)\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

$$= \boxed{x - \sqrt{3}\arctan\left(\frac{x}{\sqrt{3}}\right) + C}$$