Extra Examples of Sequences

Determine **and state** whether the following sequences **converges** or **diverges**. If it converges, compute its limit. Justify your answer. Do **not** just put down a number.

1. $\left\{\cos\left(\pi + \frac{1}{n}\right)\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \cos\left(\pi + \frac{1}{n}\right) = \cos\left(\lim_{n\to\infty} \pi + \frac{1}{n}\right) = \cos(\pi + 0) = \cos\pi = \boxed{-1}\ \boxed{\text{Converges}}$$

Notice that we passed the limit inside the continuous cosine function.

2. $\left\{\frac{3n^2 - n + 2}{5n^2 + 7}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{3n^2 - n + 2}{5n^2 + 7} = \lim_{n\to\infty} \frac{3n^2 - n + 2}{5n^2 + 7} \cdot \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty} \frac{3 - \frac{1}{n} + \frac{2}{n^2}}{5 + \frac{7}{n^2}} = \boxed{\frac{3}{5}}\ \boxed{\text{Converges}}$$

Or L'H Rule can also be used, but must switch to x variable.

3. $\left\{\sqrt{\frac{4n^2 + 5}{9n^2 - 2n + 1}}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \sqrt{\frac{4n^2 + 5}{9n^2 - 2n + 1}} = \sqrt{\lim_{n\to\infty} \frac{4n^2 + 5}{9n^2 - 2n + 1}} = \sqrt{\lim_{n\to\infty} \frac{4n^2 + 5}{9n^2 - 2n + 1} \cdot \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)}$$

$$= \sqrt{\lim_{n\to\infty} \frac{4 + \frac{5}{n^2}}{9 - \frac{2}{n} + \frac{1}{n^2}}} = \sqrt{\frac{4 + 0}{9 - 0 + 0}} = \sqrt{\frac{4}{9}} = \boxed{\frac{2}{3}}\ \boxed{\text{Converges}}$$

4. $\left\{\left(1 - \frac{3}{n^2}\right)^{n^2}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \left(1 - \frac{3}{n^2}\right)^{n^2} \overset{1^\infty}{} = \lim_{x\to\infty} \left(1 - \frac{3}{x^2}\right)^{x^2} \overset{1^\infty}{} = e^{\lim_{x\to\infty} \ln\left[\left(1 - \frac{3}{x^2}\right)^{x^2}\right]} = e^{\lim_{x\to\infty} x^2 \ln\left(1 - \frac{3}{x^2}\right)^{\infty\cdot 0}}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1 - \frac{3}{x^2}\right)}{\frac{1}{x^2}}^{\frac{0}{0}}} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\left(\frac{1}{1 - \frac{3}{x^2}}\right)\left(\frac{6}{x^3}\right)}{-\frac{2}{x^3}}} = e^{\lim_{x\to\infty} \frac{1}{1 - \frac{3}{x^2}} \cdot (-3)} = \boxed{\frac{1}{e^3}}\ \boxed{\text{Converges}}$$

5. $\left\{\left(\dfrac{n}{n+1}\right)^n\right\}_{n=1}^{\infty}$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \;\; 1^\infty \;\; = \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x \;\; 1^\infty \;\; = e^{\lim\limits_{x\to\infty}\ln\left[\left(\frac{x}{x+1}\right)^x\right]} = e^{\lim\limits_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)}\;\infty\cdot 0$$

$$= e^{\lim\limits_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}}}\;\tfrac{0}{0} \overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\frac{\left(\frac{x+1}{x}\right)\left(\frac{(x+1)(1)-x(1)}{(x+1)^2}\right)}{-\frac{1}{x^2}}} = e^{\lim\limits_{x\to\infty}\frac{\left(\frac{x+1}{x}\right)\left(\frac{1}{(x+1)^2}\right)}{-\frac{1}{x^2}}}$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{-x}{x+1}\right)}\;\tfrac{-\infty}{\infty} \overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\left(\frac{-1}{1}\right)} = e^{-1} = \boxed{\frac{1}{e}}\ \boxed{\text{Converges}}$$

6. $\left\{\left(\dfrac{n}{n+5}\right)^{2n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty}\left(\frac{n}{n+5}\right)^{2n+1}\;(1^\infty)\;\; \lim_{x\to\infty}\left(\frac{x}{x+5}\right)^{2x+1}\;(1^\infty) = e^{\lim\limits_{x\to\infty}\ln\left[\left(\frac{x}{x+5}\right)^{2x+1}\right]}$$

$$= e^{\lim\limits_{x\to\infty}(2x+1)\ln\left(\frac{x}{x+5}\right)}\;(\infty\cdot 0) = e^{\lim\limits_{x\to\infty}\frac{\ln\left(\frac{x}{x+5}\right)}{\frac{1}{2x+1}}}\;\tfrac{0}{0} \overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\frac{\left(\frac{x+5}{x}\right)\left(\frac{(x+5)(1)-x(1)}{(x+5)^2}\right)}{-\frac{2}{(2x+1)^2}}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{\left(\frac{x+5}{x}\right)\left(\frac{5}{(x+5)^2}\right)}{-\frac{2}{(2x+1)^2}}} = e^{\lim\limits_{x\to\infty}\left(\frac{x+5}{x}\right)\left(\frac{5}{(x+5)^2}\right)\left(-\frac{(2x+1)^2}{2}\right)}$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{-5(4x^2+4x+1)}{2(x^2+5x)}\right)} = e^{\lim\limits_{x\to\infty}\left(\frac{-5(4x^2+4x+1)}{2(x^2+5x)}\right)\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)} = e^{\lim\limits_{x\to\infty}\left(\frac{-5\left(4+\frac{4}{x}+\frac{1}{x^2}\right)}{2\left(1+\frac{5}{x}\right)}\right)}$$

$$= e^{-\frac{20}{2}} = e^{-10} = \boxed{\frac{1}{e^{10}}}\ \boxed{\text{Converges}}$$

7. $\left\{\dfrac{(2n+1)!}{(2n)!}\right\}$

$$\lim_{n\to\infty}\frac{(2n+1)!}{(2n)!} = \lim_{n\to\infty}\frac{(2n+1)(2n)(2n-1)\ldots 5\cdot 4\cdot 3\cdot 2\cdot 1}{(2n)(2n-1)(2n-2)\ldots 5\cdot 4\cdot 3\cdot 2\cdot 1} = \lim_{n\to\infty} 2n+1 = \boxed{\infty}\ \boxed{\text{Diverges}}$$

$$\text{OR}\ \lim_{n\to\infty}\frac{(2n+1)!}{(2n)!} = \lim_{n\to\infty}\frac{(2n+1)(2n)!}{(2n)!} = \lim_{n\to\infty} 2n+1 = \boxed{\infty}\ \boxed{\text{Diverges}}$$