Math 121 Self-Assessment Quiz #$3B$

Answer Key

- Please see the course webpage for the answer key.

1. Compute

$$\int_{-1}^{0} x^4 \arcsin x \; dx = \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{-1}^{0} \frac{x^5}{\sqrt{1-x^2}} \; dx$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0} \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}}\cos\theta \; d\theta$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0} \frac{\sin^5\theta}{\sqrt{\cos^2\theta}}\cos\theta \; d\theta = \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0} \frac{\sin^5\theta}{\cos\theta}\cos\theta \; d\theta$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0} \sin^5\theta \; d\theta = \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0} \sin^4\theta\sin\theta \; d\theta$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} - \frac{1}{5}\int_{x=-1}^{x=0} (1-\cos^2\theta)^2\sin\theta \; d\theta = \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} + \frac{1}{5}\int_{x=-1}^{x=0} (1-w)^2 \; dw$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} + \frac{1}{5}\int_{x=-1}^{x=0} 1 - 2w^2 + w^4 \; dw$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} + \frac{1}{5}\left(w - \frac{2w^3}{3} + \frac{w^5}{5}\right)\Big|_{x=-1}^{x=0}$$

$$= \frac{x^5}{5}\arcsin x\Big|_{-1}^{0} + \frac{1}{5}\left(\cos\theta - \frac{2\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right)\Big|_{x=-1}^{x=0}$$

$$= \frac{x^5}{5}\arcsin x + \frac{\sqrt{1-x^2}}{5} - \frac{2(1-x^2)^{\frac{3}{2}}}{15} + \frac{(1-x^2)^{\frac{5}{2}}}{25}\Big|_{-1}^{0}$$

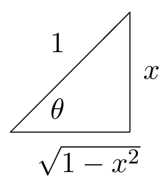
$$= 0 + \frac{1}{5} - \frac{2}{15} + \frac{1}{25} - \left(-\frac{1}{5}\arcsin(-1) + 0 - 0 + 0\right)$$

$$= \frac{1}{5} - \frac{2}{15} + \frac{1}{25} + \frac{1}{5}\left(-\frac{\pi}{2}\right) = \frac{15}{75} - \frac{10}{75} + \frac{3}{75} - \frac{\pi}{10} = \boxed{\frac{8}{75} - \frac{\pi}{10}}$$

$u = \arcsin x$	$dv = x^4 dx$
$du = \frac{1}{\sqrt{1-x^2}}dx$	$v = \frac{x^5}{5}$

Trig. Substitute

$x = \sin\theta$
$dx = \cos\theta d\theta$



Substitute $\boxed{\begin{array}{l} w = \cos\theta \\ dw = -\sin\theta d\theta \\ -dw = \sin\theta d\theta \end{array}}$

2. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(\sin^2 x + 1\right)^{\frac{7}{2}}}\ dx = \frac{43}{60\sqrt{2}}$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left[1+\sin^2 x\right]^{\frac{7}{2}}}\ dx = \int_0^1 \frac{1}{[1+u^2]^{\frac{7}{2}}}\ du$$

$$= \int_{u=0}^{u=1} \frac{1}{[1+\tan^2\theta]^{\frac{7}{2}}}\ \sec^2\theta\ d\theta = \int_{u=0}^{u=1} \frac{1}{[\sec^2\theta]^{\frac{7}{2}}}\ \sec^2\theta\ d\theta$$

$$= \int_{u=0}^{u=1} \frac{1}{\left(\sqrt{\sec^2\theta}\right)^7}\ \sec^2\theta\ d\theta = \int_{u=0}^{u=1} \frac{1}{\sec^7\theta}\ \sec^2\theta\ d\theta$$

$$= \int_{u=0}^{u=1} \frac{1}{\sec^5\theta}\ d\theta = \int_{u=0}^{u=1} \cos^5\theta\ d\theta = \int_{u=0}^{u=1} \cos^4\theta\cos\theta\ d\theta$$

$$= \int_{u=0}^{u=1} \left(\cos^2\theta\right)^2\cos\theta\ d\theta = \int_{u=0}^{u=1} \left(1-\sin^2\theta\right)^2\cos\theta\ d\theta$$

$$= \int_{u=0}^{u=1} (1-w^2)^2\ dw = \int_{u=0}^{u=1} 1 - 2w^2 + w^4\ dw = w - \frac{2w^3}{3} + \frac{w^5}{5}\Bigg|_{u=0}^{u=1}$$

$$= \sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5}\Bigg|_{u=0}^{u=1} = \frac{u}{\sqrt{u^2+1}} - \frac{2}{3}\left(\frac{u}{\sqrt{u^2+1}}\right)^3 + \frac{1}{5}\left(\frac{u}{\sqrt{u^2+1}}\right)^5\Bigg|_{u=0}^{u=1}$$

$$= \frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5 - (0 - 0 + 0)$$

$$= \frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{2\sqrt{2}}\right) + \frac{1}{5}\left(\frac{1}{4\sqrt{2}}\right) = \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}}$$

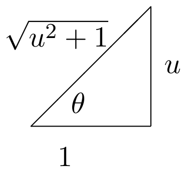
$$= \frac{60}{60\sqrt{2}} - \frac{20}{60\sqrt{2}} + \frac{3}{60\sqrt{2}} = \boxed{\frac{43}{60\sqrt{2}}}$$

Standard u substitution to simplify at the start:

$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x dx \end{array}}$ $\boxed{\begin{array}{l} x = 0 \Rightarrow u = 0 \\ x = \frac{\pi}{2} \Rightarrow u = 1 \end{array}}$

Trig. Substitute

$$\boxed{\begin{array}{l} u = \tan\theta \\[1em] du = \sec^2\theta d\theta \end{array}}$$



Standard w substitution for *odd* trig. integral $\int \cos^5\theta \; d\theta$ technique:

$$\boxed{\begin{array}{l} w = \sin\theta \\[1em] dw = \cos\theta \; d\theta \end{array}}$$

you can also change one more limit if you rather

$$\ldots = \int_0^{\frac{1}{\sqrt{2}}} (1-w^2)^2 \; dw = \ldots$$

3. Compute $\displaystyle\int \frac{1}{(x^2+4)^2} \; dx =$

$$\int \frac{1}{(x^2+4)^2} \; dx = \int \frac{1}{(4\tan^2\theta+4)^2} \; 2\sec^2\theta \; d\theta$$

$$= \int \frac{1}{(4\sec^2\theta)^2} \; 2\sec^2\theta \; d\theta = \int \frac{1}{16\sec^4\theta} \; 2\sec^2\theta \; d\theta = \frac{1}{8}\int \frac{\sec^2\theta}{\sec^4\theta} \; d\theta$$

$$= \frac{1}{8}\int \frac{1}{\sec^2\theta} \; d\theta = \frac{1}{8}\int \cos^2\theta \; d\theta = \frac{1}{8}\int \frac{1+\cos(2\theta)}{2} \; d\theta$$

$$= \frac{1}{16}\int 1+\cos(2\theta) \; d\theta = \frac{1}{16}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

$$= \frac{1}{16}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + C = \frac{1}{16}\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \frac{x}{\sqrt{x^2+4}}\left(\frac{2}{\sqrt{x^2+4}}\right)\right) + C = \boxed{\frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right) + C}$$

Trig. Substitute

$$\boxed{\begin{array}{l} x = 2\tan\theta \\[1em] dx = 2\sec^2\theta d\theta \end{array}}$$