$= 2 \sin \theta$, giving $du = 2 \cos \theta \, d\theta$ and $\sqrt{4 - u^2} = 2 \cos \theta$,

$$\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx = \int \frac{2 \sin \theta - 1}{2 \cos \theta}\, 2 \cos \theta \, d\theta$$

$$= \int (2 \sin \theta - 1)\, d\theta$$

$$= -2 \cos \theta - \theta + C$$

$$= -\sqrt{4 - u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x + 1}{2}\right) + C$$

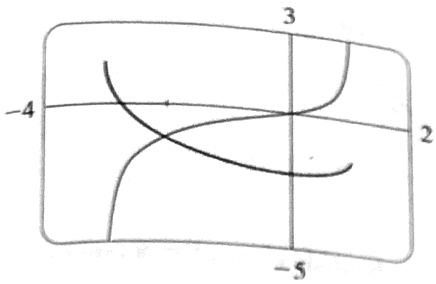

FIGURE 5

7.3 EXERCISES

1–3 Evaluate the integral using the indicated trigonometric substitution. Sketch and label the associated right triangle.

1. $\displaystyle\int \frac{dx}{x^2\sqrt{4 - x^2}}$ $\quad x = 2 \sin \theta$

2. $\displaystyle\int \frac{x^3}{\sqrt{x^2 + 4}}\,dx$ $\quad x = 2 \tan \theta$

3. $\displaystyle\int \frac{\sqrt{x^2 - 4}}{x}\,dx$ $\quad x = 2 \sec \theta$

4–30 Evaluate the integral.

4. $\displaystyle\int \frac{x^2}{\sqrt{9 - x^2}}\,dx$

5. $\displaystyle\int \frac{\sqrt{x^2 - 1}}{x^4}\,dx$

6. $\displaystyle\int_0^3 \frac{x}{\sqrt{36 - x^2}}\,dx$

7. $\displaystyle\int_0^a \frac{dx}{(a^2 + x^2)^{3/2}}$, $\quad a > 0$

8. $\displaystyle\int \frac{dt}{t^2\sqrt{t^2 - 16}}$

9. $\displaystyle\int_2^3 \frac{dx}{(x^2 - 1)^{3/2}}$

10. $\displaystyle\int_0^{2/3} \sqrt{4 - 9x^2}\,dx$

11. $\displaystyle\int_0^{1/2} x\sqrt{1 - 4x^2}\,dx$

12. $\displaystyle\int_0^2 \frac{dt}{\sqrt{4 + t^2}}$

13. $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x^3}\,dx$

14. $\displaystyle\int_0^1 \frac{dx}{(x^2 + 1)^2}$

15. $\displaystyle\int_0^a x^2\sqrt{a^2 - x^2}\,dx$

16. $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2 - 1}}$

17. $\displaystyle\int \frac{x}{\sqrt{x^2 - 7}}\,dx$

18. $\displaystyle\int \frac{d}{[(ax)^2}$ [illegible]

19. $\displaystyle\int \frac{\sqrt{1 + x^2}}{x}\,dx$

20. $\displaystyle\int \frac{t}{\sqrt{1 +}}$ [illegible]

21. $\displaystyle\int_0^{0.6} \frac{x^2}{\sqrt{9 - 25x^2}}\,dx$

22. $\displaystyle\int_0 \sqrt{x^2}$ [illegible]

23. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$

24. $\displaystyle\int_0 \sqrt{x}$ [illegible]

25. $\displaystyle\int x^2\sqrt{3 + 2x - x^2}\,dx$

26. [illegible]

27. $\displaystyle\int \sqrt{x^2 + 2x}\,dx$

28. [illegible]

29. $\displaystyle\int x\sqrt{1 - x^4}\,dx$

30. [illegible]

31. (a) Use trigonometric substitution to

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2}$$

(b) Use the hyperbolic substitution

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}($$

These formulas are connected by F

7.4 EXERCISES

1–6 Write out the form of the partial fraction decomposition of the function (as in Example 7). Do not determine the numerical values of the coefficients.

1. (a) $\dfrac{4+x}{(1+2x)(3-x)}$ (b) $\dfrac{1-x}{x^3+x^4}$

2. (a) $\dfrac{x-6}{x^2+x-6}$ (b) $\dfrac{x^2}{x^2+x+6}$

3. (a) $\dfrac{1}{x^2+x^4}$ (b) $\dfrac{x^3+1}{x^3-3x^2+2x}$

4. (a) $\dfrac{x^4-2x^3+x^2+2x-1}{x^2-2x+1}$ (b) $\dfrac{x^2-1}{x^3+x^2+x}$

5. (a) $\dfrac{x^6}{x^2-4}$ (b) $\dfrac{x^4}{(x^2-x+1)(x^2+2)^2}$

6. (a) $\dfrac{t^6+1}{t^6+t^3}$ (b) $\dfrac{x^5+1}{(x^2-x)(x^4+2x^2+1)}$

7–38 Evaluate the integral.

7. $\displaystyle\int \frac{x^4}{x-1}\,dx$

8. $\displaystyle\int \frac{3t-2}{t+1}\,dt$

9. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$

10. $\displaystyle\int \frac{y}{(y+4)(2y-1)}\,dy$

11. $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx$

12. $\displaystyle\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

13. $\displaystyle\int \frac{ax}{x^2-bx}\,dx$

14. $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx$

15. $\displaystyle\int_{-1}^0 \frac{x^3-4x+1}{x^2-3x+2}\,dx$

16. $\displaystyle\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx$

17. $\displaystyle\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$

18. $\displaystyle\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,dx$

19. $\displaystyle\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$

20. $\displaystyle\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,dx$

21. $\displaystyle\int \frac{dt}{(t^2-1)^2}$

22. $\displaystyle\int \frac{x^4+9x^2+x+2}{x^2+9}\,dx$

23. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

24. $\displaystyle\int \frac{x^2-x+6}{x^3+3x}\,dx$

31. $\displaystyle\int \frac{1}{x^3-1}\,dx$

33. $\displaystyle\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}\,dx$

35. $\displaystyle\int \frac{5x^4+7x^2+x+2}{x(x^2+1)^2}$

37. $\displaystyle\int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx$

39–52 Make a substitutio
function and then evaluate

39. $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$

41. $\displaystyle\int \frac{dx}{x^2+x\sqrt{x}}$

43. $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$

45. $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$ [

46. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$

47. $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}$ c

49. $\displaystyle\int \frac{\sec^2 t}{\tan^2 t+3\tan t}$

51. $\displaystyle\int \frac{dx}{1+e^x}$

53–54 Use integratio
this section, to evalua

53. $\displaystyle\int \ln(x^2-x+2$

55. Use a graph of f
$\int_0^2 f(x)\,dx$ is posi
gh estimate o

functions that we have been dealing with in this book are called **elementary functions**. These are the polynomials, rational functions, power functions (x^a), exponential functions (b^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cosh x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by

$$F(x) = \int_0^x e^{t^2}\,dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the functions we know. (In Chapter 11, however, we will see how to express $\int e^{x^2}\,dx$ as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3 + 1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives. You may be assured, though, that the integrals in the following exercises are all elementary functions.

7.5 EXERCISES

1–82 Evaluate the integral.

1. $\displaystyle\int \frac{\cos x}{1 - \sin x}\,dx$

2. $\displaystyle\int_0^1 (3x + 1)^{\sqrt{2}}\,dx$

3. $\displaystyle\int_1^4 \sqrt{y}\,\ln y\,dy$

4. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx$

5. $\displaystyle\int \frac{t}{t^4 + 2}\,dt$

6. $\displaystyle\int_0^1 \frac{x}{(2x + 1)^3}\,dx$

7. $\displaystyle\int_{-1}^1 \frac{e^{\arctan y}}{1 + y^2}\,dy$

8. $\displaystyle\int t \sin t \cos t\,dt$

9. $\displaystyle\int_2^4 \frac{x + 2}{x^2 + 3x - 4}\,dx$

10. $\displaystyle\int \frac{\cos(1/x)}{x^3}\,dx$

11. $\displaystyle\int \frac{1}{x^3\sqrt{x^2 - 1}}\,dx$

12. $\displaystyle\int \frac{2x - 3}{x^3 + 3x}\,dx$

13. $\displaystyle\int \sin^5 t \cos^4 t\,dt$

14. $\displaystyle\int \ln(1 + x^2)\,dx$

15. $\displaystyle\int x \sec x \tan x\,dx$

16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1 - x^2}}\,dx$

17. $\displaystyle\int_0^{\pi} t \cos^2 t\,dt$

18. $\displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

19. $\displaystyle\int e^{x + e^x}\,dx$

20. $\displaystyle\int e^2\,dx$

21. $\displaystyle\int \arctan\sqrt{x}\,dx$

22. $\displaystyle\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\,dx$

23. $\int_0^1 (1 + \sqrt{x})^8\, dx$

24. $\int (1 + \tan x)^2 \sec x\, dx$

25. $\int_0^1 \frac{1 + 12t}{1 + 3t}\, dt$

26. $\int_0^1 \frac{3x^2 + 1}{x^3 + x^2 + x + 1}\, dx$

27. $\int \frac{dx}{1 + e^x}$

28. $\int \sin\sqrt{at}\, dt$

29. $\int \ln\left(x + \sqrt{x^2 - 1}\right) dx$

30. $\int_{-1}^{2} |e^x - 1|\, dx$

31. $\int \sqrt{\frac{1 + x}{1 - x}}\, dx$

32. $\int_1^3 \frac{e^{3/x}}{x^2}\, dx$

33. $\int \sqrt{3 - 2x - x^2}\, dx$

34. $\int_{\pi/4}^{\pi/2} \frac{1 + 4\cot x}{4 - \cot x}\, dx$

35. $\int_{-\pi/2}^{\pi/2} \frac{x}{1 + \cos^2 x}\, dx$

36. $\int \frac{1 + \sin x}{1 + \cos x}\, dx$

37. $\int_0^{\pi/4} \tan^3\theta \sec^2\theta\, d\theta$

38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta \cot\theta}{\sec\theta}\, d\theta$

39. $\int \frac{\sec\theta \tan\theta}{\sec^2\theta - \sec\theta}\, d\theta$

40. $\int_0^{\pi} \sin 6x \cos 3x\, dx$

41. $\int \theta \tan^2\theta\, d\theta$

42. $\int \frac{\tan^{-1} x}{x^2}\, dx$

43. $\int \frac{\sqrt{x}}{1 + x^3}\, dx$

44. $\int \sqrt{1 + e^x}\, dx$

45. $\int x^5 e^{-x^3}\, dx$

46. $\int \frac{(x - 1)e^x}{x^2}\, dx$

47. $\int x^3 (x - 1)^{-4}\, dx$

48. $\int_0^1 x\sqrt{2 - \sqrt{1 - x^2}}\, dx$

49. $\int \frac{1}{x\sqrt{4x + 1}}\, dx$

50. $\int \frac{1}{x^2\sqrt{4x + 1}}\, dx$

51. $\int \frac{1}{x\sqrt{4x^2 + 1}}\, dx$

52. $\int \frac{dx}{x(x^4 + 1)}$

53. $\int x^2 \sinh mx\, dx$

54. $\int (x + \sin x)^2\, dx$

55. $\int \frac{dx}{x + x\sqrt{x}}$

56. $\int \frac{dx}{\sqrt{x} + x\sqrt{x}}$

57. $\int x\sqrt[3]{x + c}\, dx$

58. $\int \frac{x \ln x}{\sqrt{x^2 - 1}}\, dx$

59. $\int \frac{dx}{x^4 - 16}$

60. $\int \frac{dx}{x^2\sqrt{4x^2 - 1}}$

61. $\int \frac{d\theta}{1 + \cos\theta}$

62. $\int \frac{d\theta}{1 + \cos^2\theta}$

63. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$

64. $\int \frac{1}{\sqrt{\sqrt{x} + 1}}\, dx$

65. $\int \frac{\sin 2x}{1 + \cos^4 x}\, dx$

66. $\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x \cos x}\, dx$

67. $\int \frac{1}{\sqrt{x + 1} + \sqrt{x}}\, dx$

68. $\int \frac{x^2}{x^6 + 3x^3 + 2}\, dx$

69. $\int_1^{\sqrt{3}} \frac{\sqrt{1 + x^2}}{x^2}\, dx$

70. $\int \frac{1}{1 + 2e^x - e^{-x}}\, dx$

71. $\int \frac{e^{2x}}{1 + e^x}\, dx$

72. $\int \frac{\ln(x + 1)}{x^2}\, dx$

73. $\int \frac{x + \arcsin x}{\sqrt{1 - x^2}}\, dx$

74. $\int \frac{4^x + 10^x}{2^x}\, dx$

75. $\int \frac{dx}{x \ln x - x}$

76. $\int \frac{x^2}{\sqrt{x^2 + 1}}\, dx$

77. $\int \frac{xe^x}{\sqrt{1 + e^x}}\, dx$

78. $\int \frac{1 + \sin x}{1 - \sin x}\, dx$

79. $\int x \sin^2 x \cos x\, dx$

80. $\int \frac{\sec x \cos 2x}{\sin x + \sec x}\, dx$

81. $\int \sqrt{1 - \sin x}\, dx$

82. $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$

83. The functions $y = e^{x^2}$ and $y = x^2 e^{x^2}$ don't have elementary antiderivatives, but $y = (2x^2 + 1)e^{x^2}$ does. Evaluate $\int (2x^2 + 1)e^{x^2}\, dx$.

84. We know that $F(x) = \int_0^x e^{e^t}\, dt$ is a continuous function by FTC1, though it is not an elementary function. The functions

$$\int \frac{e^x}{x}\, dx \quad \text{and} \quad \int \frac{1}{\ln x}\, dx$$

are not elementary either, but they can be expressed in terms of F. Evaluate the following integrals in terms of F.

(a) $\int_1^2 \frac{e^x}{x}\, dx$ (b) $\int_2^3 \frac{1}{\ln x}\, dx$

7.6 Integration Using Tables and Computer Algebra Systems

In this section we describe how to use tables and computer algebra systems to integrate functions that have elementary antiderivatives. You should bear in mind, though, that even